MAT-331: Topology and Calculus

Due: Next Class

Instructions

Begin by reflecting on the following quote from [2].

> **Here we collect the topological language and properties which are indispensable for function theory (e.g., "open", "closed", "compact"). *Too much topology at the beginning is harmful, but our program would fail without any topology at all.* There is a quotation from R. Dedekind's book *Was sind und was sollen die Zahlen* (Vieweg, Braunschweig, 1887; English trans. by W. W. Beman, *Essays in the Theory of Numbers*, Dover, New York, 1963) which is equally applicable to set-theoretic topology, even though the latter had not yet appeared on the scene in Dedekind's time: "Die größten und fruchtbarsten Fortschritte in der Mathematik und anderen Wissenschaften sind vorzugsweise durch die Schöpfung und Einführung neuer Begriffe gemacht, nachdem die häufige Wiederkehr zusammengesetzter Erscheinungen, welche von den alten Begriffen nur mühselig beherrscht werden, dazu gedrängt hat (The greatest and most fruitful progress in mathematics and other sciences is made through the creation and introduction of new concepts; those to which we are impelled by the frequent recurrence of compound phenomena which are only understood with great difficulty in the older view)." Since only metric spaces ever occur in function theory, we limit ourselves to them.**

Write your thoughts down on a piece of paper. Share your thoughts in a group. Then, in those same groups, do each of the following problems and turn in to me at the beginning of next class.

I. Sketch a proof of Theorem 1.12. Consider the following points:

- For the sake of contradiction, suppose that any two points in G can be connected by a path in G, and G is not connected. Then, there exists disjoint non-empty sets X and Y such that $G = X \cup Y$. Let $a \in X$ and $b \in Y$ and define γ as a path that connects them. Show that γ must be disconnected, which contradicts the fact that paths must be connected.
- Conversely, suppose that G is a connected open set and let $z_0 \in G$. Define A to be the set of points in G that are connected to z_0 by a path, and define $B = G \setminus A$. Now, show that A and B are both open sets. Therefore, G is the disjoint union of two open sets. If both A and B are non-empty, then G is separated, which is a contradiction. Thus, one of the sets must be empty and you can show the empty set is B.

II. Use the mean value theorem, see Theorem A.2 of [1], to prove Bernoulli's inequality, for $x > 0$:

$$(1 + x)^n \geq 1 + nx$$

for all real numbers $n \geq 1$.

Hint: Let $f(t) = (1 + t)^n$ on $[0, x]$ and apply the mean value theorem.

III. Do Problems 1.29, 1.30, and 1.33 of Chapter 1 in [1].

References

[1] M. Beck, G. Marchesi, D. Pixton, and L. Sabalka, *A First Course in Complex Analysis*, vol. 1.54, Orthogonal Publishing, Ann Arbor, MI, 2002.

[2] Reinhold Remmert, *Theory of Complex Functions*, Springer-Verlag, New York, NY, 1991.