MATH-110 **Name:** ______________________

Spring 2023

Exam 2 Solution

February 21 **Pledge:** ______________________

Each question topic and point value is recorded in the tables below. Note that this exam must be completed within the 50 minutes allotted during class. Also, you must work without any external resources, which includes no notes, calculator, nor any equivalent software. You must show an appropriate amount of work to justify your answer for each problem. If you run out of room for a given problem, you may continue your work on the back of the page. By writing your name and signing the pledge you are stating that you understand the rules outlining this exam.

Scoring Table

Question	Points	Score
1	6	
2	5	
3	10	
4	8	
5	7	
Total:	36	

Topics Table

Question	Topic
1	Limit Definition of the Derivative
2	Tangent Line
3	Derivative Rules
4	Implicit Differentation
5	Related Rates

1. (a) (2 points) State the limit definition of the derivative of $f(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) (4 points) Use the limit definition to find the derivative of $f(x) = \sqrt{x}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

2. (5 points) Find the equation of the line tangent to the graph of $f(x) = x^2 + 3x + 4$ at $(1, 8)$.

   Solution: Note that $f'(x) = 2x + 3$. Therefore, the tangent line at $(1, 8)$ has slope $f'(1) = 5$. So, the equation of the tangent line in point-slope form is

$$y - 8 = 5(x - 1).$$

3. Find the derivative of each function below. Show your work.

(a) (2 points) $f(x) = 3x + 4$.

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x) + \frac{d}{dx}(4) \\ &= 3 + 0 = 3. \end{aligned}$$

(b) (4 points) $f(x) = x^{1/3}\left(x^2 + 2x + 3\right)$.

$$\begin{aligned} f'(x) &= (x^2 + 2x + 3)\frac{d}{dx}(x^{1/3}) + x^{1/3}\frac{d}{dx}(x^2 + 2x + 3) \\ &= (x^2 + 2x + 3)\frac{1}{3}x^{-2/3} + x^{1/3}(2x + 2). \end{aligned}$$

(c) (4 points) $f(x) = \dfrac{x - 1}{\sqrt{x^2 + 1}}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{\sqrt{x^2+1}\,\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}\sqrt{x^2+1}}{(\sqrt{x^2+1})^2} \\ &= \frac{\sqrt{x^2+1} - (x-1)\frac{x}{\sqrt{x^2+1}}}{x^2+1}. \end{aligned}$$

4. Consider the implicit equation

$$x^2 + xy + y^2 = 1.$$

(a) (4 points) Find $\dfrac{dy}{dx}$.

Solution: Taking the derivative of both sides with respect to x:

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}.$$

(b) (4 points) Find where the line tangent to the graph of the given implicit equation is horizontal.

Solution: There is a horizontal tangent line when $\frac{dy}{dx} = 0$, which occurs when

$$-2x - y = 0,$$

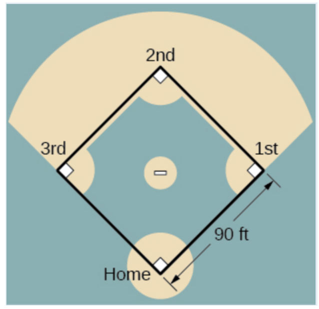
i.e., when $y = -2x$. Plugging this back into the given implicit equation implies that

$$x^2 + x(-2x) + (-2x)^2 = 1,$$

i.e., $3x^2 = 1$, so $x = \pm\frac{1}{\sqrt{3}}$. Therefore, the points on the graph of the given implicit equation where the tangent line is horizontal are

$$(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}) \text{ and } (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$$

5. (7 points) A player is running from 1st to 2nd base at a speed of 20 ft/s at the instant they are 10 ft away from 1st base. At what rate is the player's distance from home base changing at that instant.



Solution: Let x denote the distance the player is from 1st base and let y denote the distance the player is from home. Then, since y is the hypotenuse of a right triangle with sides x and 90, we have

$$x^2 + 90^2 = y^2.$$

Taking the derivative of both sides with respect to t:

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

Therefore,

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{x}{\sqrt{x^2+90^2}}\frac{dx}{dt}.$$

When $x = 10$ and $dx/dt = 20$, we have

$$\frac{dy}{dt} = \frac{10}{\sqrt{10^2+90^2}}(20)$$