· Math-140 · 2/6/2024

Chain Rule -

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Proof -

$$\frac{d}{dx} f(g(x)) = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Let $h = g(x+\Delta x) - g(x)$. Then,

$$= \lim_{h \to 0} \frac{f(g(x)+h) - f(g(x))}{h} \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(g(x)) \cdot g'(x).$$

Example -

a) $\frac{d}{dx} \cos(x^3) = \frac{d}{dx} f(g(x))$, $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $g(x) = x^3$, $g'(x) = 3x^2$

$$= -\sin(x^3) \cdot 3x^2$$

b) $\frac{d}{dx} \sin\left(\frac{1}{x}\right) = \frac{d}{dx} f(g(x))$, $f(x) = \sin(x)$, $f'(x) = \cos(x)$, $g(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$

$$= \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$

- Math-140 -

Exercises - Find the derivative of each

a) $\sin(2x)$

b) $\tan(x^2+1)$

c) $\sqrt[3]{12+\sqrt{3x}}$

d) $\cos^3\left(\frac{x}{x+1}\right)$

e) $\sqrt{x}\tan^3(\sqrt{x})$

Solution -

a) $\frac{d}{dx}\sin(2x) = \cos(2x)\cdot 2$

b) $\frac{d}{dx}\tan(x^2+1) = \sec^2(x^2+1)\cdot(2x)$

c) $\frac{d}{dx}\left(12+(3x)^{1/2}\right)^{1/3} = \frac{1}{3}\left(12+(3x)^{1/2}\right)^{-2/3}\cdot\frac{d}{dx}\left(12+(3x)^{1/2}\right)$

$= \frac{1}{3}\left(12+(3x)^{1/2}\right)^{-2/3}\cdot\frac{1}{2}(3x)^{-1/2}\cdot 3$

d) $\frac{d}{dx}\cos^3\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left[\cos\left(\frac{x}{x+1}\right)\right]^3 = 3\cos^2\left(\frac{x}{x+1}\right)\cdot\frac{d}{dx}\cos\left(\frac{x}{x+1}\right)$

$= 3\cos^2\left(\frac{x}{x+1}\right)\cdot -\sin\left(\frac{x}{x+1}\right)\frac{d}{dx}\frac{x}{x+1}$

$= -3\cos^2\left(\frac{x}{x+1}\right)\cdot\sin\left(\frac{x}{x+1}\right)\cdot\frac{1(x+1)-x(1)}{(x+1)^2}$