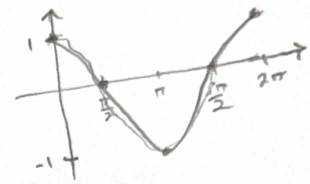
Math - 140 : 2/27/2024

- Explain why Rolle's theorem applies to $f(x) = \cos(x)$ on $[0, 2\pi]$.



<u>Solution</u> - $f(0) = f(2\pi) = 1$

- Identify the critical number $c \in (0, 2\pi)$ guaranteed by Rolle's theorem.

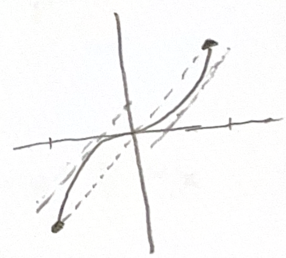
<u>Solution</u> - $f'(x) = -\sin(x) = 0$ when $x = n\pi$ { integer multiple of π

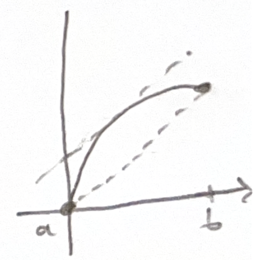
Here, $c = \pi \in (0, 2\pi)$ satisfies $f'(c) = 0$.

<u>Mean Value Theorem</u> -

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then, there is a $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

There exists a c such that the slope of the tangent line at $(c, f(c))$ is equal to the slope of the secant line from $(a, f(a))$ to $(b, f(b))$.





<u>Proof</u> -

Define $g(x) = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$

Secant line from $(a, f(a))$ to $(b, f(b))$

and apply Rolle's theorem to $h(x) = f(x) - g(x)$.

$\Rightarrow \exists\, c \in (a, b)$ s.t. $h'(c) = 0 \Rightarrow f'(c) = g'(c) = \dfrac{f(b) - f(a)}{b - a}$

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Example - Find a tangent to $f(x) = x^3$ that is parallel to the secant line from $(-2,-8)$ to $(2,8)$.

Solution

Slope of the secant line is $m = \frac{8-(-8)}{2-(-2)} = \frac{16}{4} = 4$

Hence, we want the tangent line to have slope

$$f'(x) = 3x^2 = 4$$

$$\Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$$

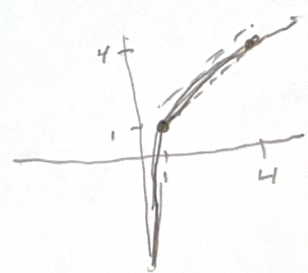
The equation of the tangent line is

$$y - \left(\frac{4}{3}\right)^{3/2} = 4\left(x - \sqrt{4/3}\right)$$

and

$$y + \left(\frac{4}{3}\right)^{3/2} = 4\left(x + \sqrt{\frac{4}{3}}\right)$$

Example - Find a tangent line to $f(x) = 5 - 4/x$ that is parallel to the secant line on $[1,4]$.



Solution -

Slope of the secant line is $m = \frac{4-1}{4-1} = 1$

Hence, we want the tangent line to have slope

$$f'(x) = 4x^{-2} = 1$$

$$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2, \quad x = 2 \text{ in interval}$$

The equation of the tangent line is

$$y - 3 = 1(x-2)$$