Extrema on an Interval

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Main Definitions and Theorems

We begin with the definition of an absolute minimum and maximum of a function.

Definition 1 (Extrema). Let $f(x)$ be defined on the interval I. Then,

I. $f(c)$ is the *absolute minimum* of f on I if $f(c) \leq f(x)$ for all x in I,

II. $f(c)$ is the *absolute maximum* of f on I if $f(c) \geq f(x)$ for all x in I.

In parts I and II of Definition I, we say that f has an absolute minimum and maximum at c, respectively. The extreme value theorem guarantees that every continuous function has an absolute minimum and maximum over a closed interval.

Theorem 1 (Extreme Value Theorem). *If f is continuous on the interval $[a, b]$, then f has an absolute minimum and maximum on the interval.*

The fact that the interval is closed plays an important role in the Extreme Value Theorem. For example, $\tan x$ is a continuous function on the open interval $(-\pi/2, \pi/2)$; however, since

$$\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty \text{ and } \lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty,$$

it follows that $\tan x$ does not have an absolute minimum nor maximum on the interval $(-\pi/2, \pi/2)$.

Next, we give the definition of a relative (local) minimum and maximum of a function.

Definition 2 (Relative Extrema).

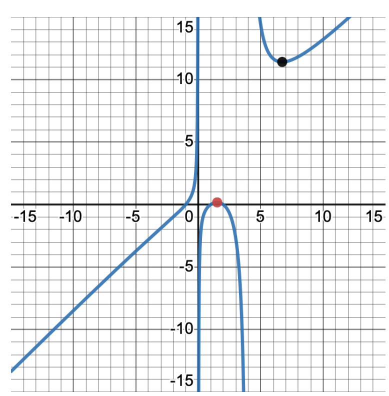
I. $f(c)$ is a *relative minimum* of f if there exists an interval I that contains c and which $f(c)$ is the absolute minimum on I.

II. $f(c)$ is a *relative maximum* of f if there exists an interval I that contains c and which $f(c)$ is the absolute maximum on I.

In parts I and II of Definition 2, we say that f has a relative minimum and maximum at c, respectively. Consider the following example of local extrema.

Example 1.

The function $f(x) = \frac{x^3-2x^2-x+2}{x^2-4x}$ is shown in the figure below. Note that $f(x)$ has no absolute maximum nor minimum over the interval $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$. However, $f(x)$ does have a relative maximum and minimum, indicated by the red and black dots, respectively. Finally, note that the relative extrema of f occur at points on the graph where the tangent line has zero slope.



In Example 1, we noted that relative extrema of the given function occur at points where the tangent line has zero slope. In what follows, we generalize this observation for all functions.

Definition 3 (Critical Number)**.** Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a *critical number* of f.

Theorem 2. *If f has a relative minimum or maximum at c, then c is a critical number of f.*

Theorem 2 motivates the following steps for finding all absolute and relative extrema of a function f.
Steps for finding all extrema:

I. Find the critical numbers of f.

II. For each critical number, determine if f has a relative minimum or maximum (or neither) by checking the value of f' at a point to the left and right of the critical number.

III. Find the value of f at the endpoints of the given interval (if the interval is closed).

IV. Compare the value of f at the endpoints of the given interval (if the interval is closed) and all relative extrema of f to determine the absolute extrema of f.

We will discuss these steps in further detail in Section 4.3. For today, please complete the following problems. Note that you are encouraged to use Desmos (or an equivalent piece of software) to help with these problems.
Problems:

I. Write down each definition and theorem from the above notes.

II. Sketch the graph of $f(x) = |x|$. Show that $c = 0$ is a critical number of f and that f has relative minimum (and absolute minimum) at $c = 0$.

III. Sketch the graph of $f(x) = x^4 - 4x^3$. Show that $c = 0$ and $c = 3$ are critical numbers of f. Then, show that f does not have a relative extrema at $c = 0$. Finally, show that f has a relative minimum (and absolute minimum) at $c = 3$.