

Homework 07 – Solutions
Math 140-002: Calculus I (Spring 2026)
Week 7

Relevant topics: related rates; linear approximations and differentials; L'Hôpital's Rule; Rolle's Theorem and the Mean Value Theorem; First Derivative Test

1. **Problem.** Evaluate $\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x}$.

Solution. Direct substitution gives $\frac{0}{0}$, so L'Hôpital's Rule applies:

$$\lim_{x \rightarrow 0} \frac{e^{5x} - 1}{x} = \lim_{x \rightarrow 0} \frac{5e^{5x}}{1} = 5.$$

2. **Problem.** Use linear approximation to estimate $\sqrt{25.3}$.

Solution. Let $f(x) = \sqrt{x}$ and choose $a = 25$. Then $f(25) = 5$ and

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(25) = \frac{1}{10}.$$

So the linearization is

$$L(x) = 5 + \frac{1}{10}(x - 25).$$

Thus

$$\sqrt{25.3} \approx L(25.3) = 5 + \frac{1}{10}(0.3) = 5.03.$$

3. **Problem.** Use differentials to estimate the change in volume of a sphere when r changes from 4 to 4.05.

Solution.

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad dV = 4\pi r^2 dr.$$

Here $r = 4$ and $dr = 0.05$, so

$$dV = 4\pi(4)^2(0.05) = 4\pi(16)(0.05) = 3.2\pi \text{ cm}^3.$$

So the volume increases by approximately $3.2\pi \text{ cm}^3$.

4. **Problem.** A circle has $\frac{dr}{dt} = 2$ cm/s. Find $\frac{dA}{dt}$ when $r = 7$.

Solution.

$$A = \pi r^2 \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

At $r = 7$ and $\frac{dr}{dt} = 2$,

$$\frac{dA}{dt} = 2\pi(7)(2) = 28\pi \text{ cm}^2/\text{s}.$$

5. **Problem.** Let $f(x) = x^3 - 3x$ on $[-2, 2]$. (a) Verify the MVT hypotheses. (b) Find all c satisfying the MVT conclusion.

Solution. (a) f is a polynomial, so it is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$.

(b) Compute the average rate of change:

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{(8 - 6) - ((-8) + 6)}{4} = \frac{2 - (-2)}{4} = \frac{4}{4} = 1.$$

Also,

$$f'(x) = 3x^2 - 3.$$

Set $f'(c) = 1$:

$$3c^2 - 3 = 1 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}}.$$

Both values lie in $(-2, 2)$.

6. **Problem.** Use the First Derivative Test to find all local extrema of $f(x) = x^{2/3}(x - 4)$.

Solution. Differentiate using the product rule:

$$f(x) = x^{2/3}(x - 4) \quad \Rightarrow \quad f'(x) = \left(\frac{2}{3}x^{-1/3}\right)(x - 4) + x^{2/3}(1).$$

Factor $x^{-1/3}$ (valid for $x \neq 0$):

$$f'(x) = x^{-1/3} \left(\frac{2}{3}(x - 4) + x\right) = x^{-1/3} \left(\frac{2}{3}x - \frac{8}{3} + x\right) = x^{-1/3} \left(\frac{5}{3}x - \frac{8}{3}\right) = \frac{1}{3}x^{-1/3}(5x - 8).$$

Critical numbers occur where $f'(x) = 0$ or DNE:

$$5x - 8 = 0 \Rightarrow x = \frac{8}{5}, \quad \text{and } f'(x) \text{ DNE at } x = 0.$$

Sign analysis:

- For $x < 0$, $x^{-1/3} < 0$ and $5x - 8 < 0$, so $f'(x) > 0$.
- For $0 < x < \frac{8}{5}$, $x^{-1/3} > 0$ and $5x - 8 < 0$, so $f'(x) < 0$.
- For $x > \frac{8}{5}$, $x^{-1/3} > 0$ and $5x - 8 > 0$, so $f'(x) > 0$.

Thus f' changes $+ \rightarrow -$ at $x = 0$, so $x = 0$ is a local maximum. Also f' changes $- \rightarrow +$ at $x = \frac{8}{5}$, so $x = \frac{8}{5}$ is a local minimum.

Values:

$$f(0) = 0, \quad f\left(\frac{8}{5}\right) = \left(\frac{8}{5}\right)^{2/3} \left(\frac{8}{5} - 4\right) = \left(\frac{8}{5}\right)^{2/3} \left(-\frac{12}{5}\right) = -\frac{12}{5} \left(\frac{8}{5}\right)^{2/3}.$$

7. **Problem.** A 15-ft ladder slides with $\frac{dx}{dt} = 1.5$ ft/s. Find $\frac{dy}{dt}$ when $x = 9$.

Solution. Let x be the distance from the wall and y the height. Then

$$x^2 + y^2 = 15^2 = 225.$$

Differentiate:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

When $x = 9$,

$$9^2 + y^2 = 225 \Rightarrow y^2 = 144 \Rightarrow y = 12.$$

So

$$\frac{dy}{dt} = -\frac{9}{12}(1.5) = -\frac{3}{4} \cdot 1.5 = -1.125 \text{ ft/s.}$$

8. **Problem.** Evaluate $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x} - x)$.

Solution. This is an $\infty - \infty$ form. Rationalize:

$$\sqrt{x^2 + 4x} - x = \frac{(\sqrt{x^2 + 4x} - x)(\sqrt{x^2 + 4x} + x)}{\sqrt{x^2 + 4x} + x} = \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Factor x from the square root:

$$\frac{4x}{\sqrt{x^2(1 + 4/x)} + x} = \frac{4x}{x\sqrt{1 + 4/x} + x} = \frac{4}{\sqrt{1 + 4/x} + 1}.$$

As $x \rightarrow \infty$, $4/x \rightarrow 0$, so

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4x} - x) = \frac{4}{1 + 1} = 2.$$

9. **Problem.** Evaluate $\lim_{x \rightarrow 0^+} (1 + 2x)^{3/x}$.

Solution. Let $y = (1 + 2x)^{3/x}$. Then

$$\ln y = \frac{3}{x} \ln(1 + 2x).$$

As $x \rightarrow 0^+$ this is $\frac{0}{0}$:

$$\ln y = 3 \cdot \frac{\ln(1 + 2x)}{x}.$$

Apply L'Hôpital's Rule to $\frac{\ln(1 + 2x)}{x}$:

$$\lim_{x \rightarrow 0^+} \frac{\ln(1 + 2x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{2}{1 + 2x}}{1} = \lim_{x \rightarrow 0^+} \frac{2}{1 + 2x} = 2.$$

Therefore

$$\lim_{x \rightarrow 0^+} \ln y = 3 \cdot 2 = 6 \quad \Rightarrow \quad \lim_{x \rightarrow 0^+} y = e^6.$$

10. **Problem.** Prove $|\sin x| \leq |x|$ for all real x .

Solution. If $x = 0$, the inequality is $0 \leq 0$, true. Assume $x \neq 0$. Apply the Mean Value Theorem to $f(t) = \sin t$ on the interval with endpoints 0 and x . Since $\sin t$ is continuous and differentiable everywhere, the MVT applies and there exists c between 0 and x such that

$$\frac{\sin x - \sin 0}{x - 0} = f'(c) = \cos c.$$

Thus

$$\frac{\sin x}{x} = \cos c.$$

Take absolute values:

$$\left| \frac{\sin x}{x} \right| = |\cos c| \leq 1.$$

Multiply both sides by $|x|$:

$$|\sin x| \leq |x|.$$

11. **Problem.** Let $g(x) = x^5 - 5x + 1$. (a) Show g has at most three real zeros. (b) Show g has exactly one real zero.

Solution. (a) Suppose (for contradiction) that g has four distinct real zeros $a < b < c < d$. By Rolle's Theorem applied on $[a, b]$, $[b, c]$, and $[c, d]$, there exist points $u \in (a, b)$, $v \in (b, c)$, $w \in (c, d)$ such that

$$g'(u) = g'(v) = g'(w) = 0.$$

Thus g' would have at least three distinct real zeros. But

$$g'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1),$$

so the real zeros of g' are $x = \pm 1$ only (two real zeros). Contradiction. Hence g has at most three real zeros.

(b) First show existence: evaluate

$$g(0) = 1 > 0, \quad g(1) = 1 - 5 + 1 = -3 < 0.$$

Since g is continuous, by the Intermediate Value Theorem there is at least one real root in $(0, 1)$.

Now show uniqueness. Compute

$$g'(x) = 5(x^4 - 1).$$

Then $g'(x) < 0$ for $|x| < 1$ and $g'(x) > 0$ for $|x| > 1$. So g is decreasing on $(-1, 1)$ and increasing on $(-\infty, -1)$ and $(1, \infty)$.

Check values at critical points:

$$g(-1) = (-1)^5 - 5(-1) + 1 = -1 + 5 + 1 = 5 > 0, \quad g(1) = -3 < 0.$$

On $(-\infty, -1)$ the function is increasing and ends at $g(-1) = 5 > 0$, so it never crosses 0 there. On $(-1, 1)$ the function is decreasing from $g(-1) = 5$ to $g(1) = -3$, so it crosses 0 exactly once in $(-1, 1)$. On $(1, \infty)$ the function is increasing starting from $g(1) = -3$. Since g is increasing and has already crossed 0 once in $(-1, 1)$, it can cross 0 at most once more on $(1, \infty)$; but we can rule this out by noting that g is increasing on $(1, \infty)$ and $g(1) = -3$, while the unique root already lies in $(0, 1)$, and g cannot return to 0 after becoming positive without creating additional critical points. Therefore there is exactly one real zero.

(Equivalently: since g is strictly decreasing on $(-1, 1)$, it has at most one zero there; since $g(-1) > 0$ and $g(1) < 0$, it has exactly one zero in $(-1, 1)$; and because g is increasing on $(1, \infty)$ with $g(1) < 0$ and g has no additional turning points, it cannot have another zero.)

12. **Problem.** Suppose f is differentiable on \mathbb{R} and $f'(x) > 0$ for all x . (a) Explain why f is one-to-one. (b) Conclude $f(x) = k$ has at most one real solution.

Solution. (a) Let $a < b$. Apply the Mean Value Theorem to f on $[a, b]$. There exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Since $b - a > 0$ and $f'(c) > 0$, we have $f(b) - f(a) > 0$, so $f(b) > f(a)$. Thus f is strictly increasing on \mathbb{R} , hence one-to-one.

(b) If $f(x) = k$ had two distinct solutions $a < b$, then $f(a) = f(b) = k$. But strict increase implies $f(b) > f(a)$, a contradiction. So $f(x) = k$ has at most one real solution.