

Math 140: Calculus I (Spring 2026)
Homework Week 10 Solutions

1. Use the First Derivative Test to find and classify all local extrema of

$$f(x) = x^3 - 6x^2 + 9x.$$

We first compute the derivative:

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3).$$

Thus the critical numbers are

$$x = 1, \quad x = 3.$$

We now make a sign chart for $f'(x)$:

x	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
$x - 1$	-	+	+
$x - 3$	-	-	+
$f'(x)$	+	-	+

So f is increasing on $(-\infty, 1)$, decreasing on $(1, 3)$, and increasing on $(3, \infty)$.

At $x = 1$, f' changes from positive to negative, so f has a local maximum at

$$f(1) = 1 - 6 + 9 = 4.$$

At $x = 3$, f' changes from negative to positive, so f has a local minimum at

$$f(3) = 27 - 54 + 27 = 0.$$

Therefore, the local extrema are

local maximum at $(1, 4)$, local minimum at $(3, 0)$.

2. Use the Second Derivative Test to find and classify all local extrema of

$$f(x) = x^4 - 4x^2 + 3.$$

First compute the first derivative:

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2).$$

Thus the critical numbers are

$$x = 0, \quad x = \pm\sqrt{2}.$$

Now compute the second derivative:

$$f''(x) = 12x^2 - 8.$$

Evaluate f'' at each critical number.

At $x = 0$:

$$f''(0) = -8 < 0,$$

so f has a local maximum at $x = 0$. Since

$$f(0) = 3,$$

the local maximum is

$$(0, 3).$$

At $x = \pm\sqrt{2}$:

$$f''(\pm\sqrt{2}) = 12(2) - 8 = 16 > 0,$$

so f has local minima at $x = \pm\sqrt{2}$. Since

$$f(\pm\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 3 = 4 - 8 + 3 = -1,$$

the local minima are

$$(-\sqrt{2}, -1) \quad \text{and} \quad (\sqrt{2}, -1).$$

Therefore,

local maximum at $(0, 3)$, local minima at $(\pm\sqrt{2}, -1)$.

3. Find the absolute maximum and absolute minimum values of

$$f(x) = x^3 - 3x + 1$$

on the closed interval $[-2, 2]$.

First compute the derivative:

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1).$$

So the critical numbers in $[-2, 2]$ are

$$x = -1, \quad x = 1.$$

Now evaluate f at the endpoints and critical numbers:

$$f(-2) = -8 + 6 + 1 = -1,$$

$$f(-1) = -1 + 3 + 1 = 3,$$

$$f(1) = 1 - 3 + 1 = -1,$$

$$f(2) = 8 - 6 + 1 = 3.$$

Thus the absolute minimum value is

$$\boxed{-1 \text{ at } x = -2 \text{ and } x = 1,}$$

and the absolute maximum value is

$$\boxed{3 \text{ at } x = -1 \text{ and } x = 2.}$$

4. Evaluate the limit

$$\lim_{x \rightarrow \infty} \frac{3x^2 + 5x - 2}{2x^2 - x + 1}.$$

Divide numerator and denominator by x^2 :

$$\lim_{x \rightarrow \infty} \frac{3 + \frac{5}{x} - \frac{2}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}} = \frac{3 + 0 - 0}{2 - 0 + 0} = \frac{3}{2}.$$

Therefore,

$$\boxed{\lim_{x \rightarrow \infty} \frac{3x^2 + 5x - 2}{2x^2 - x + 1} = \frac{3}{2}.}$$

5. Evaluate the limit using L'Hôpital's Rule

$$\lim_{x \rightarrow \infty} \frac{xe^x}{e^x + 2}.$$

As $x \rightarrow \infty$, both numerator and denominator go to ∞ , so we may apply L'Hôpital's Rule:

$$\lim_{x \rightarrow \infty} \frac{xe^x}{e^x + 2} = \lim_{x \rightarrow \infty} \frac{e^x + xe^x}{e^x} = \lim_{x \rightarrow \infty} (1 + x) = \infty.$$

Therefore,

$$\boxed{\lim_{x \rightarrow \infty} \frac{xe^x}{e^x + 2} = \infty.}$$

6. Evaluate the limit

$$\lim_{x \rightarrow \infty} (x - \ln(x + 1)).$$

Determine whether the limit diverges to ∞ , $-\infty$, or approaches a finite value.

Consider

$$x - \ln(x + 1).$$

As $x \rightarrow \infty$, the term x grows much faster than $\ln(x + 1)$. Hence

$$x - \ln(x + 1) \rightarrow \infty.$$

Therefore,

$$\boxed{\lim_{x \rightarrow \infty} (x - \ln(x + 1)) = \infty.}$$

7. Use the curve sketching checklist to analyze the function

$$f(x) = \frac{x}{x^2 + 4}.$$

Domain. Since $x^2 + 4 > 0$ for all real x , the domain is

$$(-\infty, \infty).$$

Intercepts. Since

$$\frac{x}{x^2 + 4} = 0$$

when $x = 0$, both the x - and y -intercept are

$$(0, 0).$$

Asymptotes. There are no vertical asymptotes since $x^2 + 4 \neq 0$ for all real x .

For horizontal asymptotes,

$$\lim_{x \rightarrow \infty} \frac{x}{x^2 + 4} = 0, \quad \lim_{x \rightarrow -\infty} \frac{x}{x^2 + 4} = 0.$$

So the horizontal asymptote is

$$y = 0.$$

First derivative.

$$f'(x) = \frac{(x^2 + 4) - 2x^2}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2}.$$

Critical numbers occur when $4 - x^2 = 0$, so

$$x = \pm 2.$$

Since $(x^2 + 4)^2 > 0$, the sign of $f'(x)$ is determined by $4 - x^2$:

x	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(x)$	-	+	-

Thus f is decreasing on $(-\infty, -2)$ and $(2, \infty)$, and increasing on $(-2, 2)$.

Hence there is a local minimum at

$$f(-2) = \frac{-2}{8} = -\frac{1}{4},$$

and a local maximum at

$$f(2) = \frac{2}{8} = \frac{1}{4}.$$

Second derivative. Differentiate

$$f'(x) = \frac{4 - x^2}{(x^2 + 4)^2}$$

to obtain

$$f''(x) = \frac{2x(x^2 - 12)}{(x^2 + 4)^3}.$$

Possible inflection points occur at

$$x = 0, \quad x = \pm 2\sqrt{3}.$$

Since $(x^2 + 4)^3 > 0$, the sign of $f''(x)$ is determined by $2x(x^2 - 12)$:

x	$(-\infty, -2\sqrt{3})$	$(-2\sqrt{3}, 0)$	$(0, 2\sqrt{3})$	$(2\sqrt{3}, \infty)$
$f''(x)$	-	+	-	+

Thus f is concave down on

$$(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3}),$$

and concave up on

$$(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty).$$

Inflection points occur at

$$x = -2\sqrt{3}, \quad 0, \quad 2\sqrt{3}.$$

Their coordinates are

$$f(-2\sqrt{3}) = \frac{-2\sqrt{3}}{12 + 4} = -\frac{\sqrt{3}}{8},$$

$$f(0) = 0,$$

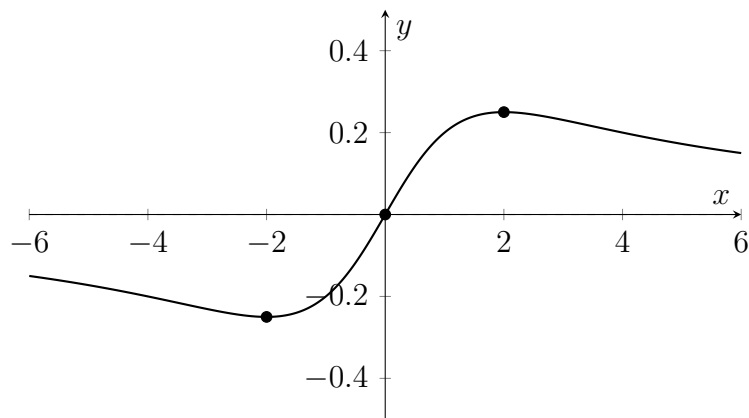
$$f(2\sqrt{3}) = \frac{2\sqrt{3}}{16} = \frac{\sqrt{3}}{8}.$$

So the inflection points are

$$\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right), \quad (0, 0), \quad \left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right).$$

Summary.

Intercept: $(0, 0)$,
horizontal asymptote: $y = 0$,
local minimum: $\left(-2, -\frac{1}{4}\right)$,
local maximum: $\left(2, \frac{1}{4}\right)$,
inflection points: $\left(-2\sqrt{3}, -\frac{\sqrt{3}}{8}\right), (0, 0), \left(2\sqrt{3}, \frac{\sqrt{3}}{8}\right)$.



8. Use the curve sketching checklist to analyze the function

$$f(x) = xe^{-x}.$$

Domain. The domain is

$$(-\infty, \infty).$$

Intercepts. Since

$$xe^{-x} = 0$$

only when $x = 0$, the only intercept is

$$(0, 0).$$

Limits at infinity.

$$\lim_{x \rightarrow \infty} xe^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$$

by L'Hôpital's Rule. Also, as $x \rightarrow -\infty$, we have $e^{-x} \rightarrow \infty$ and $x < 0$, so

$$xe^{-x} \rightarrow -\infty.$$

Thus the horizontal asymptote is

$$y = 0 \quad \text{as } x \rightarrow \infty.$$

First derivative.

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x).$$

Critical numbers occur when

$$e^{-x}(1 - x) = 0.$$

Since $e^{-x} > 0$, we get

$$x = 1.$$

Since $e^{-x} > 0$, the sign of $f'(x)$ is determined by $1 - x$:

x	$ $	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	$ $	$+$	$-$

So f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

Therefore, there is a local maximum at

$$f(1) = \frac{1}{e}.$$

Second derivative.

$$f''(x) = -e^{-x}(1 - x) - e^{-x} = e^{-x}(x - 2).$$

Possible inflection points occur at

$$x = 2.$$

Since $e^{-x} > 0$, the sign of $f''(x)$ is determined by $x - 2$:

x	$ $	$(-\infty, 2)$	$(2, \infty)$
$f''(x)$	$ $	$-$	$+$

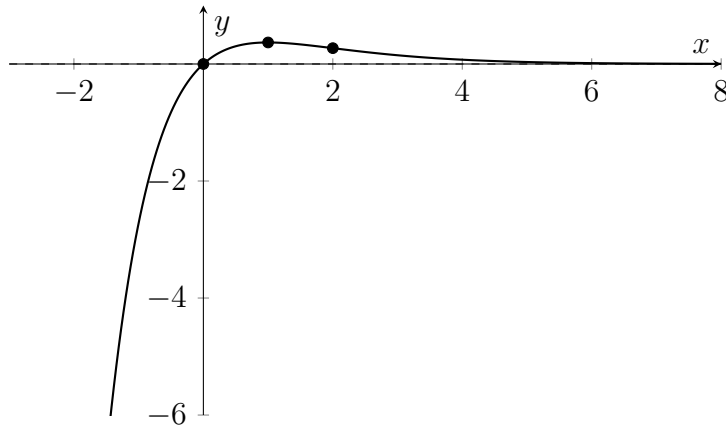
Thus the graph is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

The inflection point is

$$\left(2, \frac{2}{e^2}\right).$$

Summary.

Intercept: $(0, 0)$,
local maximum: $\left(1, \frac{1}{e}\right)$,
increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$,
concave down on $(-\infty, 2)$, concave up on $(2, \infty)$,
inflection point: $\left(2, \frac{2}{e^2}\right)$.



9. A rectangular garden is to be built along a wall using 120 meters of fencing for the other three sides.

Find the dimensions that maximize the area of the garden.

Let x be the width perpendicular to the wall, and let y be the length parallel to the wall. Then

$$2x + y = 120.$$

So

$$y = 120 - 2x.$$

The area is

$$A = xy = x(120 - 2x) = 120x - 2x^2.$$

Differentiate:

$$A'(x) = 120 - 4x.$$

Set equal to zero:

$$120 - 4x = 0 \implies x = 30.$$

Then

$$y = 120 - 2(30) = 60.$$

Therefore, the dimensions that maximize area are

30 m by 60 m.

10. A box with a square base and no top must have volume 500 cm^3 .

Find the dimensions of the box that minimize the amount of material used.

Let x be the side length of the square base, and let h be the height. Then

$$x^2h = 500,$$

so

$$h = \frac{500}{x^2}.$$

Since the box has no top, the surface area is

$$S = x^2 + 4xh.$$

Substitute for h :

$$S(x) = x^2 + 4x \left(\frac{500}{x^2} \right) = x^2 + \frac{2000}{x}.$$

Differentiate:

$$S'(x) = 2x - \frac{2000}{x^2}.$$

Set equal to zero:

$$2x - \frac{2000}{x^2} = 0.$$

Multiply by x^2 :

$$2x^3 - 2000 = 0,$$

so

$$x^3 = 1000, \quad x = 10.$$

Then

$$h = \frac{500}{10^2} = \frac{500}{100} = 5.$$

Therefore, the dimensions are

$$\boxed{10 \text{ cm} \times 10 \text{ cm} \times 5 \text{ cm}.}$$

11. A point on the curve

$$y = x^2 + 1$$

is connected to the origin by a line segment.

Find the point on the curve that minimizes the distance to the origin.

A point on the curve has coordinates

$$(x, x^2 + 1).$$

Its distance to the origin is

$$d = \sqrt{x^2 + (x^2 + 1)^2}.$$

To minimize distance, minimize

$$D(x) = x^2 + (x^2 + 1)^2.$$

Expand:

$$D(x) = x^2 + x^4 + 2x^2 + 1 = x^4 + 3x^2 + 1.$$

Differentiate:

$$D'(x) = 4x^3 + 6x = 2x(2x^2 + 3).$$

Set equal to zero:

$$2x(2x^2 + 3) = 0.$$

Since $2x^2 + 3 > 0$, the only critical number is

$$x = 0.$$

Then

$$y = 0^2 + 1 = 1.$$

Therefore, the point that minimizes distance to the origin is

$$\boxed{(0, 1)}.$$

12. Use the curve sketching checklist to analyze the function

$$f(x) = \ln(x) - \frac{x}{2}.$$

Domain. Since $\ln x$ is defined only for $x > 0$, the domain is

$$(0, \infty).$$

Intercepts. There is no y -intercept since $x = 0$ is not in the domain.

For x -intercepts, solve

$$\ln x - \frac{x}{2} = 0.$$

It is easy to check that $x = 2$ is a solution since

$$\ln 2 - 1 \neq 0,$$

so $x = 2$ is not a solution. Also $x = e$ gives

$$1 - \frac{e}{2} \neq 0.$$

This equation does not factor nicely, so we leave the intercepts described implicitly as solutions of

$$\ln x = \frac{x}{2}.$$

Since the function has a single maximum and goes to $-\infty$ at both ends of its domain, there are two x -intercepts.

Limits at infinity.

$$\lim_{x \rightarrow 0^+} \left(\ln x - \frac{x}{2} \right) = -\infty,$$

and

$$\lim_{x \rightarrow \infty} \left(\ln x - \frac{x}{2} \right) = -\infty.$$

First derivative.

$$f'(x) = \frac{1}{x} - \frac{1}{2}.$$

Critical numbers occur when

$$\frac{1}{x} - \frac{1}{2} = 0.$$

Thus

$$\frac{1}{x} = \frac{1}{2} \implies x = 2.$$

The sign chart for $f'(x)$ is

x	$(0, 2)$	$(2, \infty)$
$f'(x)$	$+$	$-$

So f is increasing on $(0, 2)$ and decreasing on $(2, \infty)$.

Therefore, there is a local maximum at

$$f(2) = \ln 2 - 1.$$

Second derivative.

$$f''(x) = -\frac{1}{x^2} < 0 \quad (x > 0).$$

Thus the graph is concave down on all of $(0, \infty)$, so there are no inflection points.

Summary.

domain $(0, \infty)$,
no y -intercept,
two x -intercepts satisfying $\ln x = \frac{x}{2}$,
local maximum at $(2, \ln 2 - 1)$,
increasing on $(0, 2)$, decreasing on $(2, \infty)$,
concave down on $(0, \infty)$, no inflection points.

