Quiz 10 Solutions
Math 140: Calculus I
Week 10

1. Consider the function
$$f(x) = \frac{x}{x^2+1}.$$

We analyze the function using the curve sketching checklist.

Domain.

Since
$$x^2 + 1 > 0$$
for every real number x, the denominator is never zero. Therefore, the domain is
$$(-\infty, \infty).$$

Intercepts.

To find the x-intercepts, solve
$$\frac{x}{x^2+1} = 0.$$
Since the denominator is never zero, this occurs when
$$x = 0.$$
Thus, the x-intercept is
$$(0, 0).$$
The y-intercept is
$$f(0) = \frac{0}{0^2+1} = 0,$$
so the y-intercept is also
$$(0, 0).$$

Vertical asymptotes.

Vertical asymptotes occur where the denominator is zero. Since
$$x^2 + 1 \neq 0$$
for all real x, there are no vertical asymptotes.

Horizontal asymptotes.

Compute the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1/x}{1+1/x^2} = 0,$$

and

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{1/x}{1+1/x^2} = 0.$$

Therefore, the horizontal asymptote is

$$y = 0.$$

First derivative.

Using the quotient rule,

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Critical numbers.

Critical numbers occur where $f'(x) = 0$ or where $f'(x)$ is undefined.

Since

$$(x^2+1)^2 > 0$$

for all x, the derivative is never undefined. So we solve

$$1 - x^2 = 0.$$

Thus,

$$x^2 = 1,$$

so the critical numbers are

$$x = -1, \qquad x = 1.$$

Increasing/decreasing intervals.

Since the denominator of $f'(x)$ is always positive, the sign of $f'(x)$ is determined by

$$1 - x^2.$$

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$1 - x^2$	$-$	$+$	$-$

Therefore, f is decreasing on

$$(-\infty, -1) \cup (1, \infty),$$

and increasing on

$$(-1, 1).$$

Local maxima and minima.

At $x = -1$, $f'(x)$ changes from negative to positive, so f has a local minimum there:

$$f(-1) = \frac{-1}{(-1)^2 + 1} = \frac{-1}{2}.$$

Thus, the local minimum is

$$\left(-1, -\frac{1}{2}\right).$$

At $x = 1$, $f'(x)$ changes from positive to negative, so f has a local maximum there:

$$f(1) = \frac{1}{1^2 + 1} = \frac{1}{2}.$$

Thus, the local maximum is

$$\left(1, \frac{1}{2}\right).$$

Second derivative.

Starting from

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2},$$

differentiate using the product rule:

$$f'(x) = (1 - x^2)(x^2 + 1)^{-2}.$$

Then

$$f''(x) = (-2x)(x^2 + 1)^{-2} + (1 - x^2)\big(-2\big)(x^2 + 1)^{-3}(2x).$$

Factor:

$$f''(x) = \frac{-2x(x^2 + 1) - 4x(1 - x^2)}{(x^2 + 1)^3}.$$

Simplify the numerator:

$$-2x(x^2 + 1) - 4x(1 - x^2) = -2x^3 - 2x - 4x + 4x^3 = 2x^3 - 6x = 2x(x^2 - 3).$$

Therefore,

$$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}.$$

Concavity and inflection points.

Since

$$(x^2+1)^3 > 0$$

for all x, the sign of $f''(x)$ is determined by

$$2x(x^2-3).$$

The possible sign changes occur at

$$x = -\sqrt{3}, \qquad x = 0, \qquad x = \sqrt{3}.$$

We make a sign chart:

Interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$2x(x^2-3)$	$-$	$+$	$-$	$+$

Thus, the graph is concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3}),$$

and concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty).$$

Since the concavity changes at all three values, the inflection points occur at

$$x = -\sqrt{3}, \qquad x = 0, \qquad x = \sqrt{3}.$$

Now compute the corresponding y-values:

$$f(-\sqrt{3}) = \frac{-\sqrt{3}}{3+1} = -\frac{\sqrt{3}}{4}, \qquad f(0) = 0, \qquad f(\sqrt{3}) = \frac{\sqrt{3}}{3+1} = \frac{\sqrt{3}}{4}.$$

So the inflection points are

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \qquad (0,0), \qquad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).$$

Summary.

The function has domain $(-\infty, \infty)$, intercept $(0,0)$, no vertical asymptotes, and horizontal asymptote $y = 0$. It is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$. It has a local minimum at

$$\left(-1, -\frac{1}{2}\right)$$

and a local maximum at

$$\left(1, \frac{1}{2}\right).$$

It is concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

and concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty),$$

with inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0,0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).$$

Sketch of the graph.