Absolute Extrema

Math 140: Calculus with Analytic Geometry

1 Review: Local Extrema

Recall that a function can have **local** (or **relative**) maxima and minima.

Definition 1.1. Let f be defined on an interval containing c.

- f has a **local maximum** at c if $f(c) \geq f(x)$ for all x near c.
- f has a **local minimum** at c if $f(c) \leq f(x)$ for all x near c.

Local extrema describe behavior near a point. They do not necessarily describe the largest or smallest value of the function on an entire interval.

Critical Numbers

Definition 1.2. A number c in the domain of f is called a **critical number** if

$$f'(c) = 0 \qquad \text{or} \qquad f'(c) \text{ does not exist.}$$

Critical numbers are candidates for local extrema.

2 Review: Classifying Critical Points

We now recall two useful tools for classifying critical points.

2.1 First Derivative Test

Theorem 2.1 (First Derivative Test). Let c be a critical number of f, and suppose f is continuous near c.

- If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.

- If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.
- If $f'(x)$ does not change sign at c, then f has no local extremum at c.

2.2 Second Derivative Test

Theorem 2.2 (Second Derivative Test). Suppose $f'(c) = 0$ and $f''(c)$ exists.

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, then the test is inconclusive.

The second derivative test is often faster, but it may fail. The first derivative test is more robust because it uses sign changes of f' directly.

3 Absolute Extrema

In many problems, we are interested in the largest and smallest values of a function on an interval.

Definition 3.1.

- A function f has an **absolute maximum** at c if
$$f(c) \geq f(x) \qquad \text{for all } x \text{ in the domain.}$$
- A function f has an **absolute minimum** at c if
$$f(c) \leq f(x) \qquad \text{for all } x \text{ in the domain.}$$

Absolute extrema are global, not local.

4 Extreme Value Theorem

Theorem 4.1 (Extreme Value Theorem). If f is continuous on a closed interval $[a, b]$, then f attains both

- an absolute maximum, and
- an absolute minimum

on $[a, b]$.

This theorem guarantees that the largest and smallest values exist.

5 Procedure on a Closed Interval

To find absolute extrema of a continuous function on $[a, b]$:

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number.
3. Evaluate $f(a)$ and $f(b)$.
4. Compare all values.

The largest value is the absolute maximum, and the smallest value is the absolute minimum.

Important idea. Even if a point is not a local maximum or minimum, an endpoint of an interval can still be an absolute extremum. That is why endpoints must always be checked on a closed interval.

6 Examples on Closed Intervals

Example 5.1 (Polynomial)

Find the absolute extrema of

$$f(x) = x^3 - 3x^2 + 2$$

on the interval $[-1, 3]$.
Solution.
First compute the derivative:

$$f'(x) = 3x^2 - 6x = 3x(x - 2).$$

The critical numbers are

$$x = 0, \qquad x = 2.$$

Now evaluate the function at the critical numbers and the endpoints:

$$f(-1) = (-1)^3 - 3(-1)^2 + 2 = -1 - 3 + 2 = -2,$$

$$f(0) = 2,$$

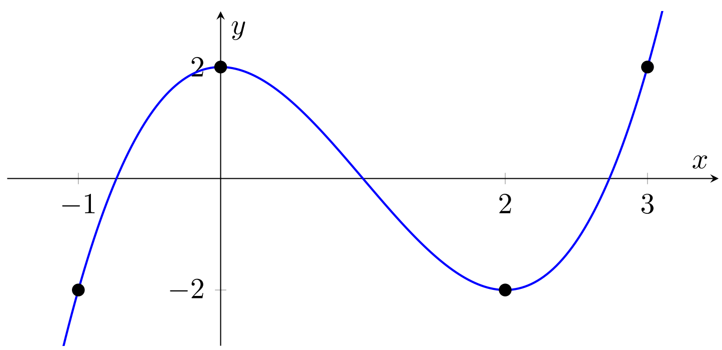
$$f(2) = 8 - 12 + 2 = -2,$$

$$f(3) = 27 - 27 + 2 = 2.$$

Comparing these values, we obtain:

$$\text{Absolute maximum: } 2 \quad \text{at } x = 0 \text{ and } x = 3,$$

$$\text{Absolute minimum: } -2 \quad \text{at } x = -1 \text{ and } x = 2.$$



Example 5.2 (Rational Function)

Find the absolute extrema of

$$f(x) = \frac{x}{x^2 + 1}$$

on the interval $[-2, 2]$.

Solution.

Differentiate using the quotient rule:

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Critical numbers occur when

$$1 - x^2 = 0.$$

Thus,

$$x = \pm 1.$$

Now evaluate the function at the critical numbers and endpoints:

$$f(-2) = \frac{-2}{4+1} = -\frac{2}{5},$$

$$f(-1) = \frac{-1}{1+1} = -\frac{1}{2},$$

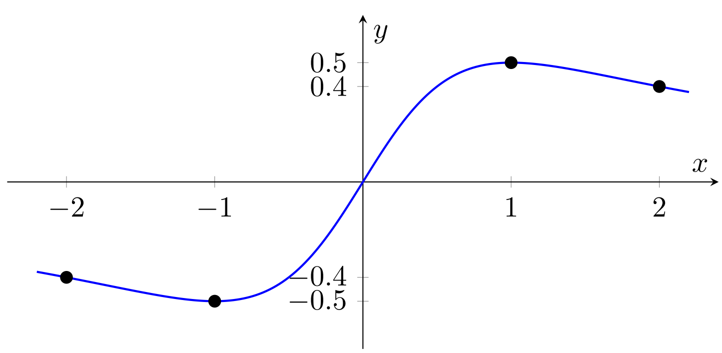
$$f(1) = \frac{1}{2},$$

$$f(2) = \frac{2}{5}.$$

Comparing these values, we conclude:

$$\text{Absolute maximum: } \frac{1}{2} \quad \text{at } x = 1,$$

$$\text{Absolute minimum: } -\frac{1}{2} \quad \text{at } x = -1.$$



Example 5.3 (Transcendental Function)

Find the absolute extrema of

$$f(x) = x + \sin x$$

on the interval $[0, 2\pi]$.
Solution.

Differentiate:

$$f'(x) = 1 + \cos x.$$

Critical numbers occur when

$$1 + \cos x = 0,$$

so

$$\cos x = -1.$$

On the interval $[0, 2\pi]$, this occurs at

$$x = \pi.$$

Now evaluate the function at the critical number and endpoints:

$$f(0) = 0 + \sin 0 = 0,$$

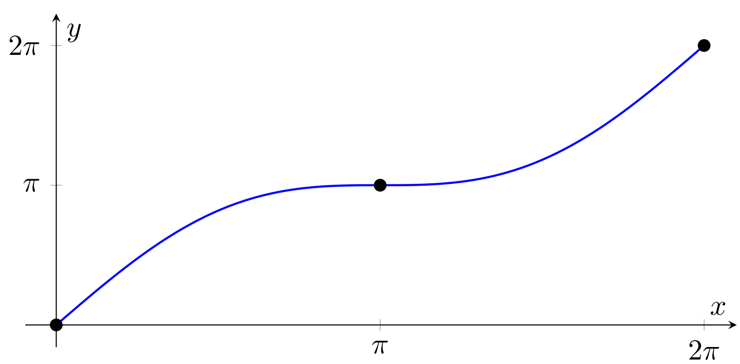
$$f(\pi) = \pi + \sin \pi = \pi,$$

$$f(2\pi) = 2\pi + \sin(2\pi) = 2\pi.$$

Comparing these values, we obtain:

$$\text{Absolute maximum: } 2\pi \quad \text{at } x = 2\pi,$$

$$\text{Absolute minimum: } 0 \quad \text{at } x = 0.$$



7 An Example on an Open Interval

The Extreme Value Theorem requires that the interval be closed. If the interval is open, absolute extrema may fail to exist.

Example 6.1 (Open Interval)

Determine whether the function

$$f(x) = x^2$$

has absolute extrema on the interval $(-1, 1)$.

Solution.

First compute the derivative:

$$f'(x) = 2x.$$

The only critical number is

$$x = 0.$$

Evaluate the function at the critical number:

$$f(0) = 0.$$

Thus, f does have an absolute minimum on $(-1, 1)$, namely

$$\text{Absolute minimum: } 0 \quad \text{at } x = 0.$$

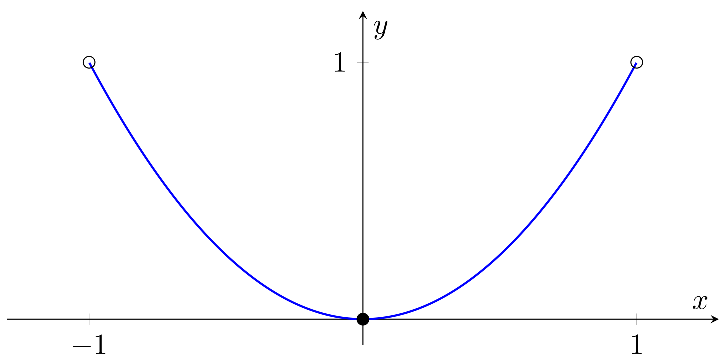
Now consider the largest possible values of $f(x)$ on $(-1, 1)$. As x approaches 1 or -1, we have

$$f(x) = x^2 \to 1.$$

However, the values $x = \pm 1$ are not in the interval. Therefore, the function never actually attains the value 1.

So f has **no absolute maximum** on $(-1, 1)$.

This example shows why the hypothesis that f is continuous on a **closed interval** $[a, b]$ is so important in the Extreme Value Theorem.



8 Comments on the Examples

These examples illustrate the main ideas:

- On a closed interval, absolute extrema can occur at critical numbers.
- On a closed interval, absolute extrema can also occur at endpoints.
- On an open interval, a function may fail to attain a largest or smallest value.

In Example 5.1, both critical numbers and endpoints contributed extrema. In Example 5.2, the extrema occurred at interior critical numbers. In Example 5.3, the extrema occurred at the endpoints, even though there was a critical number inside the interval. In Example 6.1, the interval was open, so the absolute maximum did not exist.

Thus, on a closed interval, checking only critical numbers is not enough, and on an open interval there may be no absolute maximum or minimum at all.

9 Why This Matters

Absolute extrema are central in calculus because they lead directly to optimization.

- They identify the largest and smallest possible values of a function.
- They connect local derivative information to global behavior on an interval.

- They form the foundation for applied maximum and minimum problems.
- They show why endpoints matter on closed intervals.
- They show why the closed-interval hypothesis in the Extreme Value Theorem is necessary.

This is one of the first places in calculus where local behavior and global behavior come together in a very clear way.