Antiderivatives

Math 140: Calculus with Analytic Geometry

1 Introduction

Up to this point, we have studied derivatives, which measure rates of change. We now ask the reverse question: Given a function f, can we find a function F whose derivative is f? This leads to the concept of an antiderivative. In particular, a function F is called an *antiderivative* of f on an interval if

$$F'(x) = f(x)$$

for all x in the interval. In other words, F is an antiderivative of f if differentiating F produces f.

Example

Find an antiderivative of

$$f(x) = 3x^2.$$

Since

$$\frac{d}{dx}(x^3) = 3x^2,$$

one antiderivative is

$$F(x) = x^3.$$

Example

Find an antiderivative of

$$f(x) = \cos x.$$

Since

$$\frac{d}{dx}(\sin x) = \cos x,$$

one antiderivative is

$$F(x) = \sin x.$$

Example

Find an antiderivative of

$$f(x) = \frac{1}{x}.$$

Since

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x},$$

one antiderivative is

$$F(x) = \ln|x|.$$

2 Families of Antiderivatives and Indefinite Integrals

If F is an antiderivative of f, then so is $F + C$ for any constant C. This follows because

$$\frac{d}{dx}(F(x) + C) = F'(x) = f(x).$$

Thus, antiderivatives are not unique. Instead, they form a *family of functions* that differ by a constant.

We introduce notation for this family. In particular, the *indefinite integral* of $f(x)$ is the family of all antiderivatives of f and is written as

$$\int f(x)dx.$$

If F is any antiderivative of f, then

$$\int f(x)\,dx = F(x) + C.$$

Example

All antiderivatives of

$$f(x) = 2x$$

are given by

$$\int 2xdx = x^2 + C.$$

Example

Show that the functions

$$F(x) = x^3, \quad G(x) = x^3 + 5, \quad H(x) = x^3 - 2$$

belong to the same family of antiderivatives.

Compute derivatives:

$$F'(x) = 3x^2, \quad G'(x) = 3x^2, \quad H'(x) = 3x^2.$$

Thus, each function is an antiderivative of

$$f(x) = 3x^2,$$

and they differ only by a constant. Hence,

$$\int 3x^2 dx = x^3 + C.$$

3 Properties of Indefinite Integrals

There are several basic properties of indefinite integrals that can be derived from the basic properties of derivatives. For example,

$$\int 0 dx = C,$$
$$\int Kf(x)dx = K\int f(x)dx,$$
$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx,$$
$$\int x^n = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1,$$
$$\int \frac{1}{x}dx = \ln|x| + C.$$

We can easily verify these properties using basic derivative rules. For example, $\frac{d}{dx}C = 0$ for any constant C. Also, if $F(x)$ is an antiderivative of $f(x)$, then

$$\frac{d}{dx}KF(x) = K\frac{d}{dx}F(x) = Kf(x),$$

for any constant K; hence, $KF(x)$ is an antiderivative of $Kf(x)$. The fourth property listed above is known as the *power rule for integration*. Note that, by the power rule for differentiation, we have

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{n+1}{n+1}x^{(n+1)-1} + 0 = x^n.$$

In addition to the above properties, we have many integration rules that follow directly from the differentiation rules we have already covered. For example,

$$\int e^x dx = e^x + C$$
$$\int \cos(x) dx = \sin(x) + C$$
$$\int \sec^2(x) dx = \tan(x) + C$$
$$\int \frac{1}{1+x^2} dx = \arctan(x) + C$$
$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$