# Antiderivatives

Math 140: Calculus with Analytic Geometry

A derivative describes how a function changes. We now reverse that process: given the derivative, what can we recover about the original function? This question connects differentiation to integration.

## 1 Reversing Differentiation

**Definition 1.1 (Antiderivative).** A function $F$ is an *antiderivative* of $f$ on an open interval $I$ if $F$ is differentiable on $I$ and

$$F'(x) = f(x) \qquad \text{for every } x \in I.$$

Here $f$ is given and $F$ is sought. To check an antiderivative, *differentiate it*.

**Example 1.1 (Recognizing derivatives).** Find an antiderivative of each function.

(a) $f(x) = 3x^2$. Since $(x^3)' = 3x^2$, we may take $F(x) = x^3$.

(a) $f(x) = \cos x$. Since $(\sin x)' = \cos x$, take $F(x) = \sin x$.

(a) $f(x) = 1/x$. For $x > 0$, $(\ln x)' = 1/x$. For $x < 0$,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus $F(x) = \ln|x|$ works on either interval $(-\infty, 0)$ or $(0, \infty)$.

**Example 1.2 (What information is lost?).** The functions $x^2$, $x^2 + 2$, and $x^2 - 2$ all have derivative $2x$. Their graphs have the same slope at each input, but different heights. Differentiation loses the additive constant.

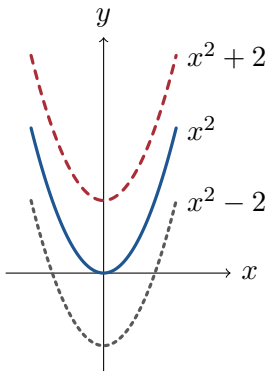



Figure 1: Vertical translations have the same derivative.

## 2 Families of Antiderivatives

Adding a constant does not change a derivative. The Mean Value Theorem tells us that, on an interval, this is the *only* freedom we have.

**Theorem 2.1 (Antiderivatives differ by a constant).** If $F$ and $G$ are antiderivatives of $f$ on an open interval $I$, then there is a constant $C$ such that $G(x) = F(x) + C$ for every $x \in I$.

*Proof.* Let $H = G - F$. Then $H' = G' - F' = f - f = 0$ on $I$. Fix $a \in I$. For any $x \in I$ with $x \neq a$, the closed interval between $a$ and $x$ lies in $I$. Because $H$ is differentiable on $I$, it is continuous on that closed interval. The Mean Value Theorem gives a point $c$ between $a$ and $x$ with

$$H(x) - H(a) = H'(c)(x - a) = 0.$$

Thus $H(x) = H(a)$ for every $x \in I$, including $x = a$. Set $C = H(a)$. □

**Definition 2.1 (Indefinite integral).** If $F' = f$ on an interval $I$, we write

$$\int f(x)\,dx = F(x) + C,$$

where $C$ is an arbitrary real constant. This notation represents the family of all antiderivatives of $f$ on $I$. The expression $f(x)$ is the *integrand*; $dx$ identifies the variable of integration.

An indefinite integral is a family of functions. It does not yet denote a number or an area. For example,

$$\int 3x^2\,dx = x^3 + C, \qquad \int 0\,dx = C, \qquad \int 1\,dx = x + C.$$

**Remark 2.1 (Why an interval is essential).** For $f(x) = 1/x$ on its full domain $\mathbb{R} \setminus \{0\}$, the constants on the two intervals can be chosen independently:

$$F(x) = \begin{cases} \ln(-x) + C_-, & x < 0, \\ \ln x + C_+, & x > 0. \end{cases}$$

Every such function has derivative $1/x$ at every point of its domain. There is no requirement that $C_- = C_+$: the Mean Value Theorem cannot be applied across 0. The familiar formula $\int 1/x\,dx = \ln|x| + C$ describes all antiderivatives on either one of the two intervals.

**One constant is enough.** Integrating $2x + 3$ gives $x^2 + 3x + C$. Any constants introduced for the separate terms combine into one arbitrary constant.

**Key idea.** State an interval on which the integrand and proposed antiderivative are defined. Verify the answer by differentiation, and include $C$ when describing all antiderivatives.

# 3 Basic Antiderivative Rules

Every rule below comes from a differentiation rule, read in reverse. Use each formula on an interval where the expressions are defined.

**Theorem 3.1 (Linearity and the power rule).** Suppose $F' = f$ and $G' = g$ on an interval, and $a, b$ are constants. Then

$$\int \big(af(x) + bg(x)\big)\,dx = aF(x) + bG(x) + C.$$

For a real number $n \neq -1$, the power rule is

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

For arbitrary real exponents, the rule always applies on $(0, \infty)$; it also applies on other open intervals where the powers are real and the differentiation rule holds.

Indeed, $(aF + bG + C)' = af + bg$, and

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{n+1}{n+1}x^n = x^n.$$

The exponent $n = -1$ is excluded because $n + 1 = 0$. Its separate rule is

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad (x \neq 0).$$

**Example 3.1 (Powers, radicals, and reciprocals).** Find all antiderivatives of $f(x) = 6x^2 - 4\sqrt{x} + 3/x^2$ on $(0, \infty)$.

$$\begin{aligned}\int \left(6x^2 - 4x^{1/2} + 3x^{-2}\right)\,dx &= 6\frac{x^3}{3} - 4\frac{x^{3/2}}{3/2} + 3\frac{x^{-1}}{-1} + C \\ &= 2x^3 - \frac{8}{3}x^{3/2} - \frac{3}{x} + C.\end{aligned}$$

Check: the derivative is $6x^2 - 4x^{1/2} + 3x^{-2}$, as required.

**Example 3.2 (Simplify before integrating).** On $(0, \infty)$, divide each term by $x$ first:

$$\int \frac{(x+1)^2}{x}\,dx = \int \left(x + 2 + \frac{1}{x}\right)\,dx = \frac{x^2}{2} + 2x + \ln x + C.$$

On $(-\infty, 0)$ the same calculation uses $\ln|x|$ instead.

**Common error.** The power rule increases the exponent by one and then *divides* by the new exponent. Linearity applies to sums and constant multiples; there is no general rule that integrates a product or quotient by integrating its factors separately.

# 4 Further Basic Rules

Recall the derivative of each function on the right to verify these formulas:

$$\int e^x\,dx = e^x + C, \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,$$
$$\int \cos x\,dx = \sin x + C, \qquad \int \sin x\,dx = -\cos x + C,$$
$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \csc^2 x\,dx = -\cot x + C,$$
$$\int \sec x\tan x\,dx = \sec x + C, \quad \int \csc x\cot x\,dx = -\csc x + C.$$

For $a^x$, assume $a > 0$ and $a \neq 1$. Trigonometric formulas hold on intervals avoiding the points where their expressions are undefined. Two further rules are

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C, \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

The first holds on $\mathbb{R}$; the second holds on $(-1, 1)$.

**Example 4.1 (Combining familiar rules).**

$$\int \left(2e^x - 3\cos x + \frac{4}{1+x^2}\right)\,dx = 2e^x - 3\sin x + 4\arctan x + C.$$

Differentiating the answer recovers all three terms of the integrand.

**Example 4.2 (Recognizing the chain rule in reverse).** Find $\int \cos(3x)\,dx$. Since

$$\frac{d}{dx}\sin(3x) = 3\cos(3x),$$

we compensate for the extra factor of 3:

$$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C.$$

Likewise, $\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$.

More generally, if $F' = f$ and $k \neq 0$ is constant, then

$$\int f(kx+b)\,dx = \frac{1}{k}F(kx+b) + C,$$

on an interval where the composition is defined. The chain rule verifies the factor $1/k$.

**Common error.** A varying inner derivative cannot be treated as a constant. For example, $\sin(x^2)$ is not an antiderivative of $\cos(x^2)$, because its derivative is $2x\cos(x^2)$. We will develop further integration methods later.

# 5 Recovering a Function from Additional Information

Knowing $F'$ determines $F$ up to an additive constant on an interval. One prescribed value of $F$ selects exactly one member of that family.

**Example 5.1 (An initial-value problem).** Find $F$ if $F'(x) = 3x^2 - 4x + 1$ on $\mathbb{R}$ and $F(1) = 2$. First find the family:

$$F(x) = x^3 - 2x^2 + x + C.$$

The condition gives $2 = 1 - 2 + 1 + C = C$. Hence

$$F(x) = x^3 - 2x^2 + x + 2.$$

Check both requirements: $F'(x) = 3x^2 - 4x + 1$ and $F(1) = 2$.

**Example 5.2 (Two successive antiderivatives).** Find $y$ if $y''(x) = 6x - 4$, $y'(0) = 3$, and $y(0) = 1$. Integrating once and using the slope condition gives

$$y'(x) = 3x^2 - 4x + C_1, \qquad 3 = y'(0) = C_1.$$

Integrating again gives

$$y(x) = x^3 - 2x^2 + 3x + C_2, \qquad 1 = y(0) = C_2.$$

Therefore $y(x) = x^3 - 2x^2 + 3x + 1$. Its second derivative and both prescribed values agree with the data. Each integration introduces its own constant.

**Example 5.3 (A tangent line determines the constant).** A curve $y = F(x)$ has $F'(x) = 2x - 2$ and tangent line $y = 2x + 1$ at $x = 2$. Find $F$.

An antiderivative gives $F(x) = x^2 - 2x + C$. The tangent line passes through $(2, F(2))$, so $F(2) = 2(2) + 1 = 5$, giving $C = 5$. Thus

$$F(x) = x^2 - 2x + 5.$$

The slope check is also necessary: $F'(2) = 2$, agreeing with the given line.

**Example 5.4 (An initial value on a restricted domain).** Find $F$ on $(0, \infty)$ if $F'(x) = 2/x$ and $F(1) = 3$. The family is $F(x) = 2\ln x + C$. Since $\ln 1 = 0$, the condition gives $C = 3$, so $F(x) = 2\ln x + 3$. A value prescribed at $x = -1$ could not determine this constant on $(0, \infty)$: that input lies outside the stated interval.

**Remark 5.1.** An initial value fixes height; the derivative fixes slopes. These are different kinds of information. If the tangent line in the last example had slope 3 at $x = 2$, no choice of $C$ could make it agree with the prescribed derivative.

# 6 Reading the Shape of an Antiderivative

The derivative tests now help us understand an unknown function $F$ from its derivative $f$:

- On an interval where $f > 0$, $F$ is increasing; on an interval where $f < 0$, $F$ is decreasing.
- At an interior point, a sign change of $f$ from positive to negative gives a local maximum of $F$; a change from negative to positive gives a local minimum.
- If $f$ is differentiable, then $F'' = f'$. Thus $f' > 0$ gives concave up behavior for $F$, and $f' < 0$ gives concave down behavior.

**Example 6.1 (From the graph of $f$ to the graph of $F$).** Suppose $F'(x) = f(x) = x^2 - 1$ on $\mathbb{R}$ and $F(0) = 0$. Then

$$F(x) = \frac{x^3}{3} - x + C, \qquad C = 0.$$

Since $f(x) = (x-1)(x+1)$, $F$ increases on $(-\infty, -1)$, decreases on $(-1, 1)$, and increases on $(1, \infty)$. Consequently,

$$F(-1) = \frac{2}{3} \text{ is a local maximum,} \qquad F(1) = -\frac{2}{3} \text{ is a local minimum.}$$

Also $F''(x) = f'(x) = 2x$, so $F$ is concave down for $x < 0$ and concave up for $x > 0$, with inflection point $(0, 0)$.

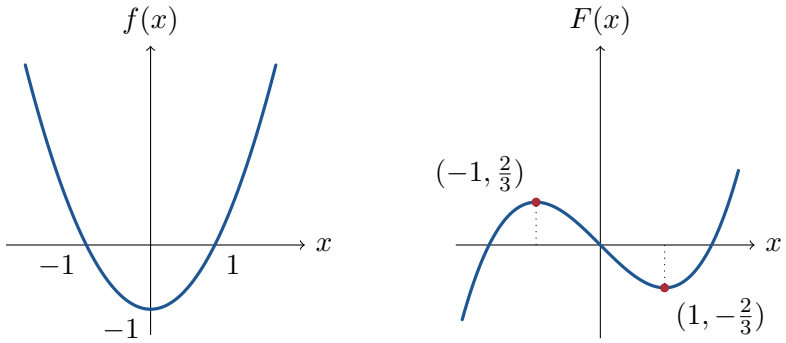



Figure 2: The sign of $f$ determines monotonicity of $F$; the slope of $f$ determines concavity of $F$.

**Example 6.2 (A zero derivative need not give an extremum).** If $F'(x) = x^2$ and $F(0) = 0$, then $F(x) = x^3/3$. Although $F'(0) = 0$, the derivative is positive on both sides of 0. Hence $F$ has no local extremum there. It does have an inflection point at $(0, 0)$, since $F''(x) = 2x$ changes sign.

**Key idea.** Antidifferentiation recovers a function from its rate of change. Differentiate to verify a formula; use the sign of the derivative to verify its shape. An additive constant changes the height of a graph, but not its monotonicity, concavity, or the inputs where extrema occur.