Approximating Area

Math 140: Calculus with Analytic Geometry

1 Introduction

In the previous lecture, we introduced the definite integral and interpreted it as the area under a curve. We also saw that when a region is bounded by a curve, we can approximate its area using rectangles.

In this lecture, we develop this idea more carefully by constructing general formulas for these approximations. We consider the area under the curve $y = x^2$ on the interval $[0, 2]$. That is, we study the quantity

$$\int_0^2 x^2 \, dx.$$

We approximate this area by dividing the interval into smaller subintervals and constructing rectangles.

2 Right Endpoint Approximation

We divide the interval $[0, 2]$ into n subintervals of equal width. The width of each subinterval is

$$\Delta x = \frac{2}{n}.$$

The right endpoints of the subintervals are

$$x_i = \frac{2i}{n}, \quad i = 1, 2, \ldots, n.$$

We construct rectangles whose heights are given by the function values at these right endpoints. Thus, the approximate area is given by

$$A_n^{\text{right}} = \sum_{i=1}^{n} f(x_i) \, \Delta x.$$

Since $f(x) = x^2$, we obtain

$$A_n^{\text{right}} = \sum_{i=1}^{n} \left(\frac{2i}{n} \right)^2 \cdot \frac{2}{n}.$$

Simplifying, we get

$$A_n^{\text{right}} = \sum_{i=1}^{n} \frac{4i^2}{n^2} \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3} \sum_{i=1}^{n} i^2.$$

Example

For $n = 8$, we divide $[0, 2]$ into eight subintervals of length $\Delta x = 1/4$ and use right endpoints. The approximate area is given by

$$A_8^{\text{right}} = \frac{8}{8^3} \cdot \sum_{i=1}^{8} i^2 = \frac{8}{512} \cdot 204 = \frac{1632}{512} \approx 3.19.$$

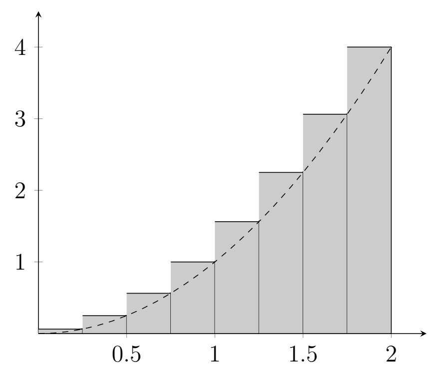

Figure 1: Right endpoint approximation with $n = 8$.

3 Left Endpoint Approximation

We again divide the interval into n equal subintervals with width $\Delta x = \frac{2}{n}$. The left endpoints are

$$x_i = \frac{2(i-1)}{n}, \quad i = 1, 2, \ldots, n.$$

The approximate area is

$$A_n^{\text{left}} = \sum_{i=1}^{n} f(x_i)\, \Delta x.$$

Substituting $f(x) = x^2$, we obtain

$$A_n^{\text{left}} = \sum_{i=1}^{n} \left(\frac{2(i-1)}{n} \right)^2 \cdot \frac{2}{n}.$$

Thus,

$$A_n^{\text{left}} = \frac{8}{n^3} \sum_{i=1}^{n} (i-1)^2.$$

Example

For $n = 8$, we divide $[0, 2]$ into eight subintervals of length $\Delta x = 1/4$ and use left endpoints. The approximate area is given by

$$A_8^{\text{left}} = \frac{8}{8^3} \cdot \sum_{i=1}^{8} (i-1)^2 = \frac{8}{512} \cdot 140 = \frac{1120}{512} \approx 2.19.$$

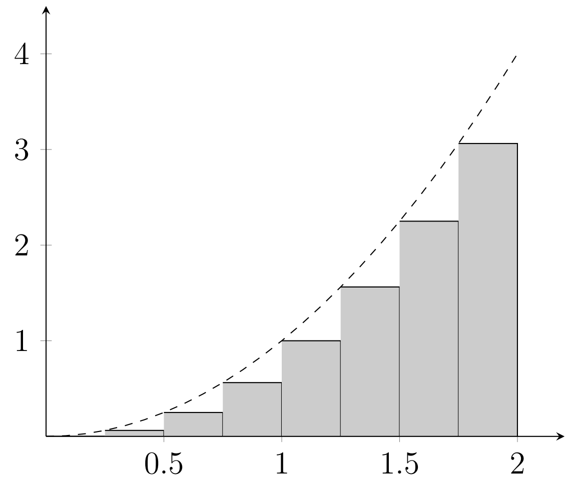

Figure 2: Left endpoint approximation with $n = 8$.

4 Midpoint Approximation

We again divide the interval into n equal subintervals of width $\Delta x = \frac{2}{n}$. The midpoint of each subinterval is

$$x_i = \frac{2i-1}{n}, \quad i = 1, 2, \ldots, n.$$

The approximate area is

$$A_n^{\text{mid}} = \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

Substituting $f(x) = x^2$, we obtain

$$A_n^{\text{mid}} = \sum_{i=1}^{n} \left(\frac{2i-1}{n}\right)^2 \cdot \frac{2}{n}.$$

Thus,

$$A_n^{\text{mid}} = \frac{2}{n^3} \sum_{i=1}^{n} (2i-1)^2.$$

Example

For $n = 8$, we divide $[0, 2]$ into eight subintervals of length $\Delta x = 1/4$ and use midpoints. The approximate area is given by

$$A_8^{\text{mid}} = \frac{2}{8^3} \sum_{i=1}^{8} (2i-1)^2 = \frac{2}{512} \cdot 680 = \frac{1360}{512} \approx 2.66.$$

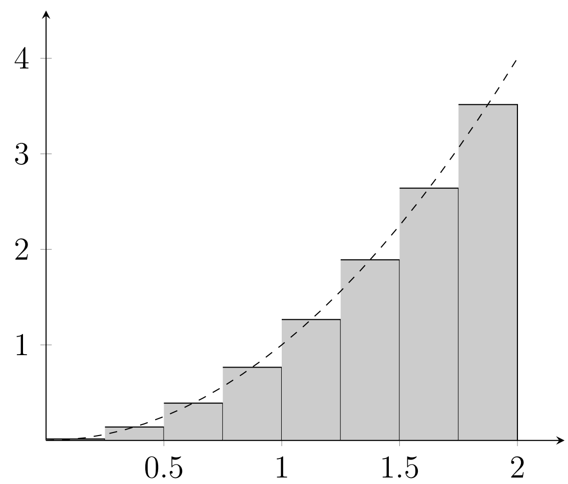

Figure 3: Midpoint approximation with $n = 8$.

5 Summary

Each of these approximations gives a different estimate of the area under the curve. As n increases, the width of each rectangle becomes smaller, and the approximation becomes more accurate. These expressions produce sequences of numbers that approach a limiting value. In the next lecture, we will study sequences and series in order to understand how these approximations behave as n becomes large. This will allow us to define the definite integral precisely using Riemann sums.