

# Approximate Area Sums

Math 140: Calculus with Analytic Geometry

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## 1 Introduction

In the previous lecture, we approximated the area under the curve  $y = x^2$  on the interval  $[0, 2]$  using left endpoint rectangles, right endpoint rectangles, and midpoint rectangles. In each case, the approximation depended on the number  $n$  of subintervals. This produces a sequence of approximations:

$$A_1, A_2, A_3, \dots$$

This sequence is an example of a sequence of partial sums. As  $n$  becomes larger, the rectangles become thinner, and the approximations become more accurate.

## 2 Sequences of Approximations

Recall that for the interval  $[0, 2]$  divided into  $n$  equal parts, the width of each subinterval is

$$\Delta x = \frac{2}{n}.$$

From the previous lecture, the right endpoint approximation is

$$A_n^{\text{right}} = \frac{8}{n^3} \sum_{i=1}^n i^2.$$

Since  $f(x) = x^2$  is increasing on  $[0, 2]$ , the right endpoint rule overestimates the area, but the overestimate becomes smaller as  $n$  increases.

Similarly, the left endpoint approximation is

$$A_n^{\text{left}} = \frac{8}{n^3} \sum_{i=1}^n (i-1)^2,$$

Because  $f(x) = x^2$  is increasing, the left endpoint rule underestimates the area.

Finally, the midpoint approximation is

$$A_n^{\text{mid}} = \frac{2}{n^3} \sum_{i=1}^n (2i-1)^2,$$

Each of these summations lead to sequence of approximations. To understand their long-term behavior, we now study the partial sums that appear in these formulas.

### 3 Summation Formulas

We begin with the easiest sum

$$\sum_{i=1}^n (1) = 1 + 1 + \cdots + 1 = n.$$

Next consider the sum

$$\begin{aligned}\sum_{i=1}^n i &= 1 + 2 + \cdots + (n-1) + n \\ &= n + (n-1) + \cdots + 2 + 1.\end{aligned}$$

Therefore,

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

For the sum of squares, note that

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1,$$

for any  $i$ . Therefore,

$$\sum_{i=1}^n ((i+1)^3 - i^3) = 3 \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n (1).$$

On the left, we have a telescoping series such that

$$\sum_{i=1}^n ((i+1)^3 - i^3) = (n+1)^3 - 1^3.$$

Therefore, we can solve for the sum of squares

$$\begin{aligned}\sum_{i=1}^n i^2 &= \frac{(n+1)^3 - 1}{3} - \sum_{i=1}^n i - \frac{1}{3} \sum_{i=1}^n (1) \\ &= \frac{n^3 + 3n^2 + 3n}{3} - \frac{n(n+1)}{2} - \frac{n}{3} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

#### 3.1 Approximate Area Formula

We are now ready to write down the exact formula for the right endpoint, left endpoint, and midpoint approximations to the integral  $\int_0^2 x^2 dx$ . For the right-endpoint rule, we have

$$\begin{aligned}A_n^{\text{right}} &= \frac{8}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

For the left-endpoint rule, we have

$$\begin{aligned} A_n^{\text{left}} &= \frac{8}{n^3} \sum_{i=1}^n (i-1)^2 \\ &= \frac{8}{n^3} \sum_{i=1}^{n-1} i^2 \\ &= \frac{8}{n^3} \cdot \frac{(n-1)n(2n-1)}{6}. \end{aligned}$$

For the midpoint rule, we have

$$\begin{aligned} A_n^{\text{mid}} &= \frac{2}{n^3} \sum_{i=1}^n (2i-1)^2 \\ &= \frac{2}{n^3} \sum_{i=1}^n (4i^2 - 4i + 1) \\ &= \frac{2}{n^3} \left( 4 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + n \right) \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^3} \cdot \frac{n(n+1)}{2} + \frac{2}{n^3} \cdot n. \end{aligned}$$

### 3.2 Limiting Values

By letting the number of rectangles grow without bound our approximations become exact. Using L'Hopital's rule or other algebraic rules, we find that

$$\lim_{n \rightarrow \infty} A_n^{\text{right}} = \lim_{n \rightarrow \infty} A_n^{\text{left}} = \lim_{n \rightarrow \infty} A_n^{\text{mid}} = \frac{8}{3}.$$

## 4 Summary

This computation is important for two reasons. First, it shows that the definite integral can be understood as the limit of a sequence of area approximations. Second, it shows that the exact area does not depend on whether we use left endpoints, right endpoints, or midpoints, provided that we let the number of rectangles increase without bound. For the function  $f(x) = x^2$  on  $[0, 2]$ , the left endpoint rule approaches the area from below, the right endpoint rule approaches it from above, and the midpoint rule gives an approximation that is already quite accurate for moderate values of  $n$ . All three lead to the same exact value:

$$\int_0^2 x^2 dx = \frac{8}{3}.$$