Approximate Area Sums II

Math 140: Calculus with Analytic Geometry

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1 Introduction

In the previous lecture, we studied the integral

$$\int_0^2 x^2\,dx$$

by writing area approximations as sequences of partial sums. We now apply the same idea to the integrals

$$\int_{-1}^1 x^2\,dx \qquad \text{and} \qquad \int_0^1 e^x\,dx.$$

In each case, we use right-endpoint rectangles and then let the number of rectangles grow without bound.

2 The Integral $\int_{-1}^1 x^2 dx$

We divide the interval $[-1, 1]$ into n equal subintervals. Since the interval has length 2, the width of each subinterval is

$$\Delta x = \frac{2}{n}.$$

The right endpoints are

$$x_i = -1 + \frac{2i}{n} = \frac{2i - n}{n}, \qquad i = 1, 2, \ldots, n.$$

Thus, the right-endpoint approximation is

$$\begin{aligned} A_n^{\text{right}} &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(\frac{2i-n}{n}\right)^2 \frac{2}{n} \\ &= \frac{2}{n^3} \sum_{i=1}^{n} (2i-n)^2. \end{aligned}$$

Now expand the square:

$$(2i - n)^2 = 4i^2 - 4ni + n^2.$$

Therefore,

$$\sum_{i=1}^{n}(2i-n)^2 = 4\sum_{i=1}^{n} i^2 - 4n\sum_{i=1}^{n} i + n^2\sum_{i=1}^{n}(1).$$

Using the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n}(1) = n,$$

we obtain

$$A_n^{\text{right}} = \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot n$$

Now, let n grow without bound. Then,

$$\lim_{n\to\infty} A_n^{\text{right}} = \frac{8}{3} - 4 + 2 = \frac{2}{3}.$$

Therefore,

$$\int_{-1}^{1} x^2\,dx = \frac{2}{3}.$$

3 The Integral $\int_0^1 e^x dx$

We divide the interval $[0,1]$ into n equal subintervals, so the width is

$$\Delta x = \frac{1}{n}.$$

The right endpoints are

$$x_i = \frac{i}{n}, \qquad i = 1, 2, \ldots, n.$$

Thus, the right-endpoint approximation is

$$\begin{aligned} A_n^{\text{right}} &= \sum_{i=1}^{n} e^{i/n}\frac{1}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(e^{1/n}\right)^i. \end{aligned}$$

This is a multiple of a geometric sum with ratio

$$r = e^{1/n}.$$

Consider the geometric sum

$$S = \sum_{i=1}^{n} r^i = r + r^2 + \cdots + r^n.$$

Then,

$$rS = r^2 + \cdots + r^n + r^{n+1}.$$

Subtracting the right hand side of each equation gives $S - rS = r - r^{n+1}$. Therefore,

$$S = \frac{r - r^{n+1}}{1 - r}.$$

In our case, $r = e^{1/n}$. Hence, we have

$$\begin{aligned} A_n^{\text{right}} &= \frac{1}{n} \cdot e^{1/n} \cdot \frac{1 - \left(e^{1/n}\right)^n}{1 - e^{1/n}} \\ &= \frac{e^{1/n}}{n} \cdot \frac{1 - e}{1 - e^{1/n}} \\ &= \frac{e^{1/n}(1 - e)}{n(1 - e^{1/n})}. \end{aligned}$$

Note that

$$\lim_{n\to\infty} e^{1/n}(1 - e) = (1 - e).$$

Also, using L'hopitals rule, we find

$$\begin{aligned} \lim_{n\to\infty} n(1 - e^{1/n}) &= \lim_{n\to\infty} \frac{1 - e^{1/n}}{1/n} \\ &= \lim_{n\to\infty} \frac{-e^{1/n}(-1/n^2)}{(-1/n^2)} \\ &= \lim_{n\to\infty} -e^{1/n} = -1. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} A_n^{\text{right}} = e - 1$$

and

$$\int_0^1 e^x \, dx = e - 1.$$