Approximation Sums

Math 140: Calculus with Analytic Geometry

In *Approximating Area*, we used a fixed number of rectangles to estimate area. Now we will find formulas that work for every positive integer n and take the limit as n grows. We will use the squeeze theorem to show why the limit gives the exact value of the definite integral in our examples.

1 From a Finite Sum to a Limit

Let f be continuous and nonnegative on $[a, b]$, with $a < b$. Recall the common width $\Delta x = (b - a)/n$, the partition points $x_i = a + i\Delta x$, and the midpoints $m_i = (x_{i-1} + x_i)/2$. The three approximation sums are

$$A_n^{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad A_n^{\text{right}} = \sum_{i=1}^{n} f(x_i)\Delta x, \quad A_n^{\text{midpoint}} = \sum_{i=1}^{n} f(m_i)\Delta x.$$

For $f(x) = x^2$ on $[0, 2]$, the previous notes gave $A_4^{\text{left}} = 7/4$, $A_4^{\text{right}} = 15/4$, and $A_4^{\text{midpoint}} = 21/8$. Each is one term in a sequence of estimates. For example, $A_1^{\text{right}}, A_2^{\text{right}}, A_3^{\text{right}}, \ldots$ is the sequence we get by recomputing the right sum for each n.



Figure 1: The same region with four and eight right endpoint rectangles.

Because x^2 is increasing on $[0, 2]$, the left rectangles lie inside the region and the right rectangles cover it. For every n,

$$A_n^{\text{left}} \le \int_0^2 x^2\,dx \le A_n^{\text{right}}.$$

If the two bounding sequences have the same limit, the squeeze theorem gives the exact value of the integral. First we simplify the finite sums so that we can take their limits.

Common error. As n changes, so do the width and the sample points. For example, the first right rectangle has area $(2/n)^3$. An approximation sequence is a sequence of finite sums, with the rectangles rebuilt at each step.

2 Formulas for Finite Sums

We need formulas that replace a sum of n terms with an expression involving n alone. Throughout this section, n is a positive integer.

Theorem 2.1 (Constant, linear, and quadratic sums).

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Why the formulas work. The first sum adds n copies of 1. For the second, let $S = \sum_{i=1}^{n} i$. Write the terms in both orders and add corresponding entries:

$$S = 1 + 2 + \cdots + (n-1) + n,$$
$$S = n + (n-1) + \cdots + 2 + 1.$$

Each of the n pairs has sum $n+1$, so $2S = n(n+1)$.

For the third formula, expand $(i+1)^3 - i^3 = 3i^2 + 3i + 1$ and sum from $i = 1$ to n. The left side is a *telescoping sum*. Each intermediate cube appears once with each sign:

$$(2^3 - 1^3) + (3^3 - 2^3) + \cdots + ((n+1)^3 - n^3) = (n+1)^3 - 1.$$

Using the first two formulas gives

$$(n+1)^3 - 1 = 3\sum_{i=1}^{n} i^2 + 3\,\frac{n(n+1)}{2} + n.$$

Solving for the sum of squares and factoring yields

$$\sum_{i=1}^{n} i^2 = \frac{(n+1)^3 - 1}{3} - \frac{n(n+1)}{2} - \frac{n}{3} = \frac{n(n+1)(2n+1)}{6}.$$

Example 2.1 (Expand the summand first). Use the sum laws to simplify $\sum_{i=1}^{n}(2i-1)^2$:

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= 4\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + n \\ &= \frac{n(4n^2-1)}{3}.\end{aligned}$$

For $n = 2$, the sum is $1^2 + 3^2 = 10$, in agreement with the formula.

Key idea. For a fixed n, any factor depending only on n can be pulled outside a sum over i. The quantity $\Delta x = (b-a)/n$ is one such factor.

3 Endpoint Sums and Exact Integrals

Example 3.1 (Evaluate $\int_0^2 x^2\,dx$ using endpoint sums). Use n equal subintervals, so $\Delta x = 2/n$ and $x_i = 2i/n$. For the right endpoint sum, substitute the sample point and multiply by the width:

$$\begin{aligned} A_n^{\text{right}} &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \end{aligned}$$

The left sum uses $x_{i-1} = 2(i-1)/n$. One way to simplify it is to write $\sum_{i=1}^{n}(i-1)^2 = 0^2 + 1^2 + \cdots + (n-1)^2$. We can also get it directly from the difference between the two sums.

Theorem 3.1 (Endpoint gap). On an equal-width partition of $[a, b]$,

$$A_n^{\text{right}} - A_n^{\text{left}} = \Delta x \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{b-a}{n}[f(b) - f(a)].$$

Every interior value occurs in both sums, leaving only $f(b) - f(a)$.

For x^2 on $[0, 2]$, the gap is $(2/n)(4-0) = 8/n$. Thus

$$A_n^{\text{left}} = A_n^{\text{right}} - \frac{8}{n} = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.$$

The terms containing $1/n$ and $1/n^2$ tend to zero. The bounds from Section 1 and the squeeze theorem now give

$$\int_0^2 x^2\,dx = \lim_{n\to\infty} A_n^{\text{left}} = \lim_{n\to\infty} A_n^{\text{right}} = \frac{8}{3}.$$

Key idea. For a continuous, nonnegative function that is increasing or decreasing on $[a, b]$, the two endpoint sums bound the integral. For an increasing function the left sum is the lower bound. For a decreasing function the right sum is the lower bound. In either case, the distance between the sums is

$$\left|A_n^{\text{right}} - A_n^{\text{left}}\right| = \frac{b-a}{n}|f(b) - f(a)|.$$

This distance tends to zero. Once we find the limit of one endpoint sum, the other endpoint sum has the same limit, and the squeeze theorem gives the value of the integral.

4 Midpoint Sums and the Same Exact Value

Example 4.1 (The midpoint sum for $y = x^2$ on $[0, 2]$). The midpoint of $[x_{i-1}, x_i]$ is $m_i = (2i - 1)/n$. Substituting and using the formula from Section 2 gives

$$A_n^{\text{midpoint}} = \sum_{i=1}^{n} \left(\frac{2i-1}{n}\right)^2 \frac{2}{n} = \frac{2}{n^3} \sum_{i=1}^{n} (2i-1)^2$$
$$= \frac{2}{n^3} \cdot \frac{n(4n^2-1)}{3} = \frac{8}{3} - \frac{2}{3n^2}.$$

Taking the limit gives

$$\lim_{n\to\infty} A_n^{\text{midpoint}} = \lim_{n\to\infty} \left(\frac{8}{3} - \frac{2}{3n^2}\right) = \frac{8}{3} = \int_0^2 x^2\, dx.$$

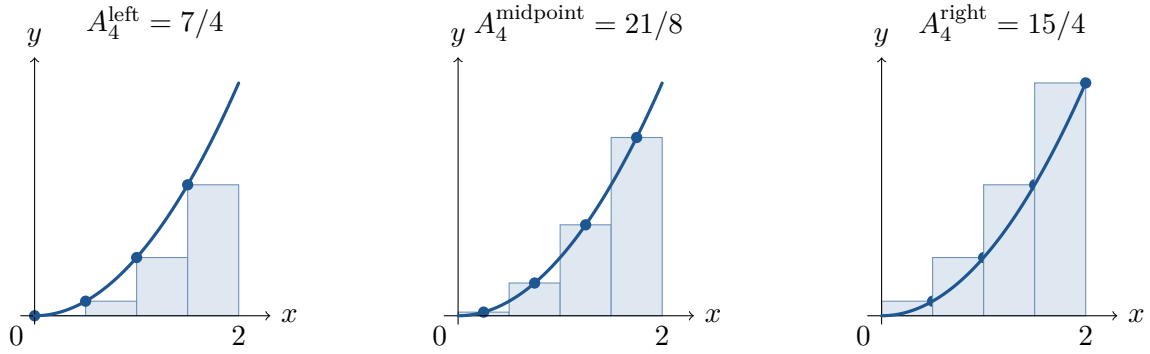


Figure 2: Three different sums for $n = 4$ approach the same integral as n grows.

The three formulas show how each finite sum differs from the common limit:

$$A_n^{\text{left}} = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}, \qquad A_n^{\text{midpoint}} = \frac{8}{3} - \frac{2}{3n^2}, \qquad A_n^{\text{right}} = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.$$

Every correction term tends to zero. The sampling rule changes the finite sum, while the limit is $8/3$ in all three cases.

Common error. Take the limit of the whole sum. Each rectangle gets thinner as n grows, and the number of rectangles also grows. For $f(x) = 1$, each rectangle contributes $(b - a)/n$, so all n rectangles together contribute $b - a$ for every n.

5 Evaluating Constant, Linear, and Quadratic Integrals

Example 5.1 (A constant integrand: $f(x) = 3$ on $[0, 2]$). With $\Delta x = 2/n$, both endpoint sums equal $\frac{6}{n}\sum_{i=1}^{n} 1 = 6$ for every n. Thus

$$\int_0^2 3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} 3\frac{2}{n} = 6.$$

Example 5.2 (A linear integrand: $f(x) = x$ on $[0, 2]$). The right sample points are $x_i = 2i/n$, so

$$A_n^{\text{right}} = \sum_{i=1}^{n} \frac{2i}{n}\frac{2}{n} = \frac{4}{n^2}\sum_{i=1}^{n} i = 2 + \frac{2}{n}.$$

Since x is increasing, the endpoint gap argument gives $\int_0^2 x\,dx = \lim_{n\to\infty}(2 + 2/n) = 2$.

Example 5.3 (Evaluate $\int_1^3 x^2\,dx$ using right endpoint sums). *Set up the partition.* The interval has length 2, so the width is again $2/n$. Its starting point is now 1, giving $x_i = 1 + 2i/n$.

Simplify the finite sum. Use the width and square the entire sample point:

$$\begin{aligned} A_n^{\text{right}} &= \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^2 \\ &= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= 2 + 4\frac{n+1}{n} + \frac{4}{3}\frac{(n+1)(2n+1)}{n^2}. \end{aligned}$$

Take the limit. Since x^2 is increasing on $[1, 3]$, the endpoint gap argument gives

$$\int_1^3 x^2\,dx = \lim_{n\to\infty} A_n^{\text{right}} = 2 + 4 + \frac{8}{3} = \frac{26}{3}.$$

6 Cubic Sums and Combinations

Example 6.1 (A cubic integrand: $f(x) = x^3$ on $[0, 2]$). With $\Delta x = 2/n$ and $x_i = 2i/n$, the right endpoint sum is

$$A_n^{\text{right}} = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \frac{2}{n} = \frac{16}{n^4} \sum_{i=1}^{n} i^3.$$

To simplify the sum, set $T_i = i(i+1)/2$ for $i = 0, 1, \ldots, n$. Since $T_i - T_{i-1} = i$ and $T_i + T_{i-1} = i^2$, we have

$$T_i^2 - T_{i-1}^2 = (T_i - T_{i-1})(T_i + T_{i-1}) = i^3.$$

Summing telescopes, leaving $\sum_{i=1}^{n} i^3 = T_n^2 - T_0^2 = [n(n+1)/2]^2$. Therefore

$$A_n^{\text{right}} = \frac{16}{n^4} \left(\frac{n(n+1)}{2}\right)^2 = 4\left(1 + \frac{1}{n}\right)^2.$$

Since x^3 is increasing on $[0, 2]$, the right endpoint limit gives

$$\int_0^2 x^3 \, dx = \lim_{n\to\infty} 4\left(1 + \frac{1}{n}\right)^2 = 4.$$

Example 6.2 (Combine finite sums to evaluate $\int_0^2 (x^3 + x + 3)\, dx$). Use the same right sample points $x_i = 2i/n$ for all three terms. By the finite sum laws,

$$\begin{aligned} A_n^{\text{right}} &= \sum_{i=1}^{n} x_i^3 \Delta x + \sum_{i=1}^{n} x_i \Delta x + \sum_{i=1}^{n} 3\Delta x \\ &= 4\left(1 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right) + 6. \end{aligned}$$

The last line uses the finite formulas from the preceding examples. The integrand is increasing, so

$$\int_0^2 (x^3 + x + 3)\, dx = \lim_{n\to\infty} A_n^{\text{right}} = 4 + 2 + 6 = 12.$$

Key idea. These examples used equal-width subintervals and left endpoints, right endpoints, or midpoints. Next we will allow different widths and other choices of sample points, leading to the definition of the Riemann integral.