Area and the Definite Integral

Math 140: Calculus with Analytic Geometry

1 Introduction

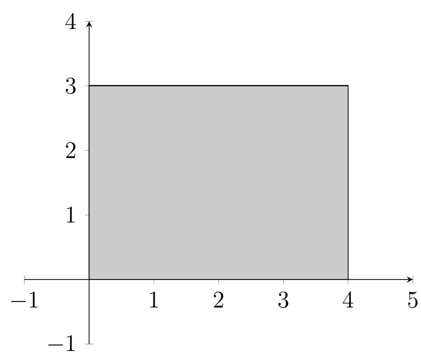
In this lecture, we study the concept of area and the definite integral. We begin with regions whose areas can be computed using familiar geometric formulas. We then consider regions bounded by curves where these formulas no longer apply. This will motivate a new approach to computing area.

2 Area Using Geometry

We begin by reviewing how to compute areas of simple regions.

Example

Find the area of the region bounded by $y = 3$, $x = 0$, $x = 4$, and the x-axis.



This region is a rectangle with base 4 and height 3, so the area is $A = 12$.

Example

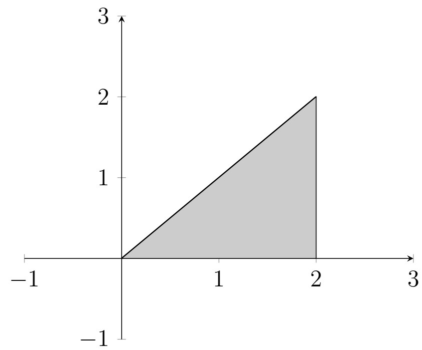
Find the area of the region bounded by $y = x$, $x = 0$, $x = 2$, and the x-axis.

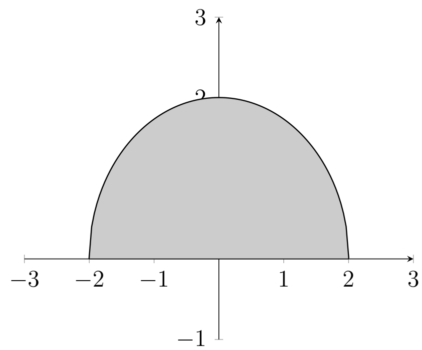
This region forms a right triangle with base 2 and height 2. Thus, the area is $A = 2$.

Example

Find the area of the region bounded by $y = \sqrt{4 - x^2}$ and the x-axis.

This is the upper half of a circle of radius 2, so the area is $A = 2\pi$.





Example

Find the area of the region bounded by $y = 2 - x$, $x = 0$, $x = 1$, and the x-axis.

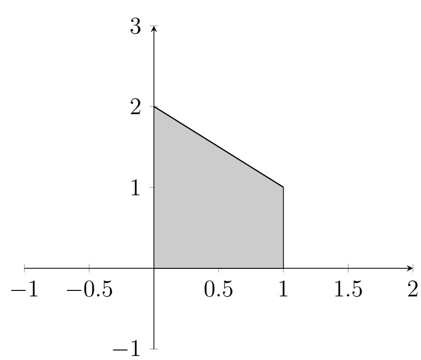
This region can be broken up into a 1×1 square and a triangle of height 1 and width 1. Therefore, the area is $A = 1 + \frac{1}{2} = \frac{3}{2}$.

3 The Definite Integral

The *definite integral* is denoted by

$$\int_a^b f(x)dx$$

and represents the area bounded by the $y = f(x)$, $x = a$, $x = b$, and the x-axis. Hence, each of the geometry problems we previously solved corresponds to a definite integral. In



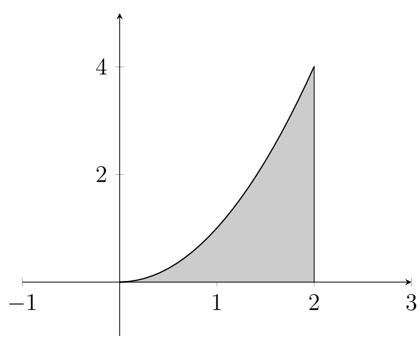
particular, we have

$$\int_0^4 3dx = 12,$$
$$\int_0^2 xdx = 2,$$
$$\int_{-2}^2 \sqrt{4-x^2}dx = 2\pi,$$
$$\int_0^1 (2-x)dx = \frac{3}{2}.$$

Often times the region formed by a curve will have area that cannot be computed via basic geometric formula. In this case, we consider approximations to the area.

Example

The area represented by $\int_0^2 x^2dx$ is shown in the figure below.

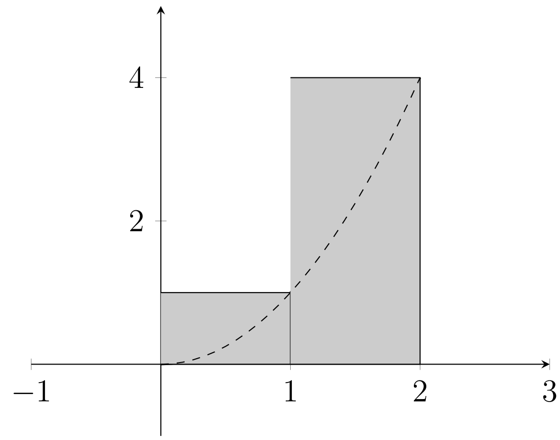


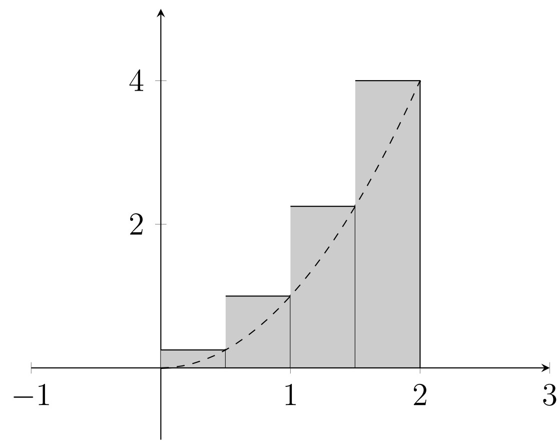
We approximate the area of the shaded region using rectangles. We begin with two rectangles as shown in the figure below. Note that each rectange has width of 1 and a height that corresponds to the value of $y = x^2$ at the right endpoint. The area approximation has a value of

$$\int_0^2 x^2dx \approx 1 \cdot 1 + 1 \cdot 4 = 5.$$

We can improve our approximation by using more rectangles. The figure below uses four rectangles to approximate the area under the curve. Note that each rectangle has a width of 1/2 and a height that corresponds to the value of $y = x^2$ at the right endpoint. The area approximation has a value of

$$\int_0^2 x^2dx \approx \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{9}{4} + \frac{1}{2} \cdot 4 = \frac{15}{4}.$$





4 Summary

We have seen that areas of simple regions can be computed using geometric formulas. However, for regions bounded by curves, we approximate the area using rectangles. As the number of rectangles increases, the approximation improves. This leads naturally to the definition of the definite integral as a limit of these approximations.