Basic Derivative Rules

Math 140: Calculus with Analytic Geometry

Key Topics

- Constant and power rules
- Constant multiple and sum/difference rules
- Differentiating polynomials efficiently
- Interpreting derivatives as slopes and rates of change

1 Why We Need Derivative Rules

The limit definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

allows us to compute derivatives from first principles, but repeating this limit calculation for every new function quickly becomes impractical. In this lecture we develop a small collection of derivative rules that let us compute derivatives efficiently.

2 Constant and Power Rules

Theorem 2.1 (Constant Rule)**.** *If $f(x) = c$, where c is a constant, then $f'(x) = 0$.*

Proof. Fix a. Using the limit definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

Since a was arbitrary, $f'(x) = 0$ for all x. □

Theorem 2.2 (Power Rule for Positive Integers)**.** *If $f(x) = x^n$ where n is a positive integer, then*

$$f'(x) = nx^{n-1}.$$

Proof. Fix a and let $f(x) = x^n$. Then

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^n - a^n}{h}.$$

By the binomial theorem,

$$(a+h)^n = a^n + \binom{n}{1} a^{n-1} h + \binom{n}{2} a^{n-2} h^2 + \cdots + h^n.$$

Subtract a^n and factor out h:

$$(a+h)^n - a^n = h \left(\binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} h + \cdots + h^{n-1} \right).$$

Therefore,

$$\frac{(a+h)^n - a^n}{h} = \binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} h + \cdots + h^{n-1}.$$

Taking $h \to 0$ gives

$$f'(a) = \binom{n}{1} a^{n-1} = n a^{n-1}.$$

Since a was arbitrary, $f'(x) = n x^{n-1}$. □

Example 2.1. *Differentiate* $f(x) = x^5$*.*

By the power rule,

$$f'(x) = 5x^4.$$

Example 2.2. *Differentiate* $f(x) = \sqrt[3]{x} = x^{1/3}$*.*

We will later justify the power rule for negative and rational exponents using the quotient rule and implicit differentiation. A complete justification for arbitrary real exponents will be given after logarithmic differentiation is introduced. For now, we record the expected result:

$$\frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3}.$$

3 Linearity Rules

The next two rules explain how derivatives behave with respect to addition and constant multiples.

Theorem 3.1 (Constant Multiple Rule)**.** *If* $f(x) = c\,g(x)$ *where* c *is a constant and* g *is differentiable, then*

$$f'(x) = c\,g'(x).$$

Proof. Fix a. Using the limit definition and factoring out the constant c,

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{c\,g(a+h) - c\,g(a)}{h} = \lim_{h \to 0} c \frac{g(a+h) - g(a)}{h} \\ &= c \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = c\,g'(a). \end{aligned}$$

□

Theorem 3.2 (Sum and Difference Rules)**.** *If* $f(x) = g(x) \pm h(x)$ *and both* g *and* h *are differentiable, then*

$$f'(x) = g'(x) \pm h'(x).$$

Proof. Fix a. Using the limit definition and limit laws,

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{(g(a+h) \pm h(a+h)) - (g(a) \pm h(a))}{h} \\ &= \lim_{h \to 0} \left(\frac{g(a+h) - g(a)}{h} \pm \frac{h(a+h) - h(a)}{h} \right) \\ &= \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \pm \lim_{h \to 0} \frac{h(a+h) - h(a)}{h} \\ &= g'(a) \pm h'(a). \end{aligned}$$

□

4 Differentiating Polynomials

Combining the constant rule, power rule, constant multiple rule, and sum/difference rules allows us to differentiate polynomials quickly.

Example 4.1. *Differentiate* $p(x) = 4x^6 - 3x^2 + 7x - 9$.

$$p'(x) = 24x^5 - 6x + 7.$$

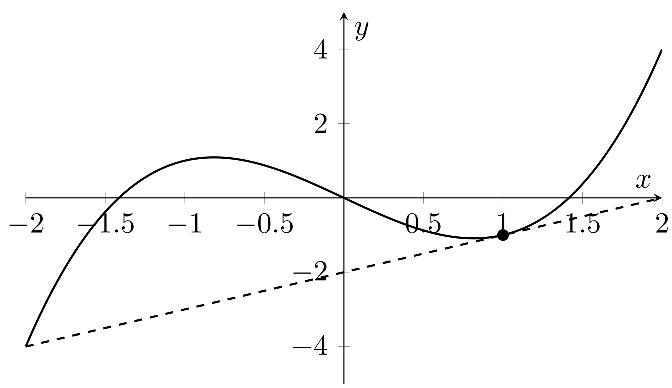
Example 4.2. *Find the equation of the tangent line to* $y = x^3 - 2x$ *at* $x = 1$.

First compute the derivative:

$$y' = 3x^2 - 2.$$

At $x = 1$, the slope is $m = 3(1)^2 - 2 = 1$. The point on the curve is $(1, 1^3 - 2 \cdot 1) = (1, -1)$. Thus the tangent line is

$$y + 1 = 1(x - 1) \implies y = x - 2.$$



5 Why This Matters for Calculus

- Derivative rules let us compute slopes and rates of change quickly without redoing limit calculations.
- The power rule and linearity rules are the foundation for differentiating polynomials and many other functions.
- Efficient differentiation is essential for optimization, graphing, and motion problems.