The Chain Rule

Math 140: Calculus with Analytic Geometry

Key Topics

- Composition of functions
- Statement and interpretation of the chain rule
- Identifying inner and outer functions
- Differentiating composite functions
- Combining the chain rule with power, product, and quotient rules
- Common pitfalls when applying the chain rule

1 Motivation

Many functions encountered in calculus are built by *composing* simpler functions. For example,

$$f(x) = \sin(x^2), \qquad g(x) = \sqrt{3x+1}, \qquad h(x) = \cos(5x).$$

These functions are not simple polynomials or basic trigonometric functions, but combinations of both. To differentiate such functions, we need a new rule.

2 Function Composition

Definition 2.1. *Let f and g be functions. The* composition *of f and g is the function*

$$(f \circ g)(x) = f\big(g(x)\big).$$

Remark 2.1. *In a composite function, $g(x)$ is often called the* inner function *and f is called the* outer function*.*

3 Statement of the Chain Rule

Theorem 3.1 (Chain Rule)**.** *If $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x, then*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Equivalently, if $y = f(g(x))$, then

$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\, g'(x).$$

4 Understanding the Chain Rule

The chain rule reflects how a change in x affects y through an intermediate variable. A small change in x first changes $u = g(x)$, which then changes $y = f(u)$.

Remark 4.1. *The derivative of a composite function is the product of the derivative of the outer function (evaluated at the inner function) and the derivative of the inner function.*

5 Basic Examples

Example 5.1. *Differentiate* $f(x) = (3x + 1)^5$.

Let $u = 3x + 1$. Then $f(x) = u^5$.

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 5u^4 \cdot 3 = 15(3x + 1)^4.$$

Example 5.2. *Differentiate* $g(x) = \sqrt{2x^2 + 1}$.

Write $g(x) = (2x^2 + 1)^{1/2}$. Let $u = 2x^2 + 1$.

$$g'(x) = \frac{1}{2}u^{-1/2} \cdot 4x = \frac{2x}{\sqrt{2x^2 + 1}}.$$

6 Trigonometric Examples

Example 6.1. *Differentiate* $h(x) = \sin(x^2)$.

Let $u = x^2$. Then

$$h'(x) = \cos(u) \cdot 2x = 2x\cos(x^2).$$

Example 6.2. *Differentiate* $p(x) = \cos(5x)$.

Let $u = 5x$.

$$p'(x) = -\sin(u) \cdot 5 = -5\sin(5x).$$

7 Combining the Chain Rule with Other Rules

Example 7.1. *Differentiate* $q(x) = x^2 \sin(3x)$.

Using the product rule and chain rule,

$$q'(x) = 2x\sin(3x) + x^2\cos(3x) \cdot 3 = 2x\sin(3x) + 3x^2\cos(3x).$$

Example 7.2. *Differentiate* $r(x) = \dfrac{1}{\sqrt{1 + x^2}}$.

Write $r(x) = (1 + x^2)^{-1/2}$.

$$r'(x) = -\frac{1}{2}(1 + x^2)^{-3/2} \cdot 2x = -\frac{x}{(1 + x^2)^{3/2}}.$$

8 Geometric Interpretation

The chain rule explains how the slope of a composite function depends on the slopes of its component functions.

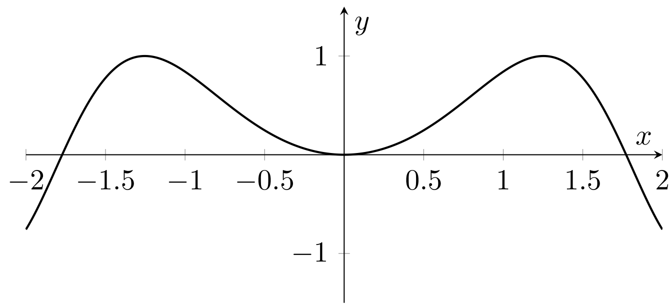

Figure 1: The graph of $y = \sin(x^2)$ illustrates how changes in x are magnified by the inner function x^2 before affecting the outer function $\sin(x)$.

9 Common Pitfalls

Remark 9.1.

- *Do not differentiate the inner function and outer function separately.*
- *Always multiply by the derivative of the inner function.*
- *Watch carefully for hidden compositions, such as powers, roots, and trigonometric expressions.*

10 Why This Matters for Calculus

The chain rule greatly expands the class of functions we can differentiate.

- It allows us to differentiate most functions encountered in practice.
- It unifies differentiation rules under a single guiding principle.
- It is essential for applications involving rates of change and related rates.
- It prepares us for implicit differentiation and logarithmic differentiation.