Applications of Derivatives Quiz: Part 1

Math 140: Calculus with Analytic Geometry

Answer Key

1. **Related rates.**

   Air leaks from a spherical balloon at a rate of 12π cubic centimeters per second. The balloon remains spherical as it shrinks. How fast is its radius changing when its radius is 3 centimeters?

   Define your variables, write an equation relating them, and differentiate with respect to time before substituting the radius. Include units and explain the sign of your answer. Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.

   Solution. Let t be time in seconds, $r = r(t)$ the radius in centimeters, and $V = V(t)$ the volume in cubic centimeters. Since air leaks out, $dV/dt = -12\pi$. Differentiating the volume equation with respect to time gives

   $$V = \frac{4}{3}\pi r^3, \qquad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

   At $r = 3$,

   $$-12\pi = 4\pi(3)^2 \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{1}{3} \text{ cm/s}.$$

   The negative sign means the radius is decreasing, at 1/3 cm/s.

2. **The first derivative test.**

Let $f(x) = x^4 - 4x^3$, with domain $\mathbb{R}$.

(a) Find $f'(x)$ and all critical numbers of f.

(b) Make a sign chart for f' and state the open intervals on which $f' > 0$ and those on which $f' < 0$. Use these signs to describe where f is increasing or decreasing.

(c) Use the *first derivative test* to classify each critical number as a local maximum, a local minimum, or neither. Give the value of each local extremum and justify each classification using the signs of f'.

Solution. The derivative is

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3).$$

It exists everywhere and is zero at 0 and 3, so these are all critical numbers.

x	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	$-$	$-$	$+$
f	decreasing	decreasing	increasing

The function is decreasing on $(-\infty, 3)$ (the isolated zero of f' at 0 does not interrupt this) and increasing on $(3, \infty)$. At 0, the derivative is negative on both sides, so there is no local extremum. At 3, it changes from negative to positive, so there is a local minimum of $f(3) = 81 - 108 = -27$. There is no local maximum.

3. **The second derivative test.**

Let $g(x) = x^2 e^{-x}$, with domain $\mathbb{R}$.

(a) Find $g'(x)$ and all critical numbers of g. Factor your derivative to make the critical numbers clear.

(b) Find $g''(x)$ and use the *second derivative test* to classify every critical number. Show the second derivative value used in each classification, and give the value of each local extremum.

Solution. The product and chain rules give

$$g'(x) = 2xe^{-x} - x^2 e^{-x} = e^{-x}x(2 - x).$$

Since $e^{-x} > 0$ for every real x, the critical numbers are 0 and 2. Differentiating again gives

$$g''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) = e^{-x}(x^2 - 4x + 2).$$

At 0, $g'(0) = 0$ and $g''(0) = 2 > 0$, so there is a local minimum of $g(0) = 0$. At 2, $g'(2) = 0$ and $g''(2) = -2e^{-2} < 0$, so there is a local maximum of $g(2) = 4e^{-2} = 4/e^2$.