Applications of Derivatives Quiz: Part 3

Math 140: Calculus with Analytic Geometry

Answer Key

Part 3: Curve Sketching and Optimization.

1. **Curve sketching.**

   Let $r(x) = \dfrac{x}{x^2 - 4}$.

   (a) State the domain. Find all intercepts and asymptotes, showing the limits that justify the asymptotes.

   (b) Compute $r'(x)$ and $r''(x)$. Use sign analysis to find intervals of increase, decrease, and concavity, and identify all local extrema and inflection points.

   (c) Sketch the graph using your results. Label intercepts, asymptotes, and inflection points, and show the behavior of every branch.

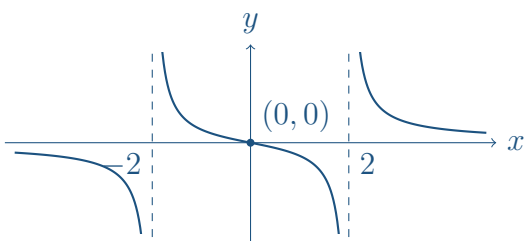
   Solution. The domain is $\mathbb{R} \setminus \{-2, 2\}$, and both intercepts are $(0, 0)$. Both end limits are 0, so $y = 0$ is a horizontal asymptote. Also,

x approaches	-2^-	-2^+	2^-	2^+
$r(x)$ approaches	$-\infty$	$+\infty$	$-\infty$	$+\infty$

   Thus $x = -2$ and $x = 2$ are vertical asymptotes; there is no slant asymptote. Differentiating and simplifying gives

   $$r'(x) = -\frac{x^2 + 4}{(x^2 - 4)^2}, \qquad r''(x) = \frac{2x(x^2 + 12)}{(x^2 - 4)^3}.$$

   The first derivative is negative throughout the domain. Hence r decreases on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, with no local extrema. The signs of r'' on $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$ are $-, +, -, +$. Thus the graph is concave down on $(-\infty, -2)$ and $(0, 2)$ and concave up on $(-2, 0)$ and $(2, \infty)$. Its only inflection point is $(0, 0)$; ± 2 are excluded from the domain.



2. **Optimization: a rectangle beneath a curve.**

A rectangle has its lower side on the x-axis and its two upper vertices on $y = \sqrt{4 - x^2}$, above the axis. Coordinates are measured in meters. Find the dimensions that maximize its area.

(a) Draw a labeled sketch and define your variables. Write a mathematical model with the area as the objective and include geometric and domain constraints.

(b) Reduce the area to a function of one variable and state its feasible domain.

(c) Use calculus to justify an *absolute* maximum. Give the rectangle's full width, height, and maximum area, including units.

Solution. The upper vertices have the same height and thus opposite x-coordinates. Write them as $(-x, y)$ and (x, y), where $x, y > 0$. The width is $2x$.

$$\begin{aligned} \max_{x,\,y} \quad & A = 2xy \\ \text{s.t.} \quad & y = \sqrt{4 - x^2}, \\ & x > 0, \quad y > 0 \end{aligned}$$

Thus

$$A(x) = 2x\sqrt{4 - x^2}, \qquad 0 < x < 2.$$

Differentiating and simplifying gives

$$A'(x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} = \frac{4(2 - x^2)}{\sqrt{4 - x^2}}.$$

The only feasible critical number is $x = \sqrt{2}$. The derivative is positive on $(0, \sqrt{2})$ and negative on $(\sqrt{2}, 2)$, proving an absolute maximum. Equivalently, the continuous extension to $[0, 2]$ satisfies $A(0) = A(2) = 0$ and $A(\sqrt{2}) = 4$. The optimal full width is $2\sqrt{2}$ m, the height is $\sqrt{2}$ m, and the maximum area is 4 m^2.

3. **Optimization: the nearest points on a curve.**

Find all points on the curve $y = 1/x$ that are closest to the origin $(0, 0)$. Coordinates are measured in meters.

(a) Write a mathematical model with distance to the origin as the objective and include the curve and domain constraints.

(b) Express the *squared* distance as a function of x. Explain why minimizing squared distance is equivalent to minimizing distance, and find all critical numbers.

(c) Justify an *absolute* minimum. Give every closest point and the minimum distance, including units.

Solution. For a point (x, y) on the curve, the model is

$$\begin{aligned} \min_{x,\,y} \quad & D = \sqrt{x^2 + y^2} \\ \text{s.t.} \quad & y = 1/x, \\ & x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

The square root is strictly increasing on $[0, \infty)$, so minimizing D is equivalent to minimizing

$$F(x) = D^2 = x^2 + \frac{1}{x^2}, \qquad x \neq 0.$$

Differentiate and factor:

$$F'(x) = 2x - \frac{2}{x^3} = \frac{2(x-1)(x+1)(x^2+1)}{x^3}.$$

The critical numbers are -1 and 1; 0 is not in the domain. The derivative is negative on $(-\infty, -1)$ and $(0, 1)$ and positive on $(-1, 0)$ and $(1, \infty)$. For a global comparison across both branches,

$$F(x) = \left(x - \frac{1}{x}\right)^2 + 2 \geq 2,$$

with equality exactly when $x^2 = 1$. Thus the closest points are $(-1, -1)$ and $(1, 1)$, and the minimum distance is $\sqrt{2}$ m.