Derivatives of Exponential and Logarithmic Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- The natural exponential function e^x
- Derivative of e^x from first principles
- Derivatives of exponential functions a^x and $e^{g(x)}$
- The natural logarithm $\ln(x)$ as the inverse of e^x
- Derivatives of logarithmic functions and the chain rule
- Using implicit differentiation with exponential and logarithmic functions
- Tangent lines to exponential and logarithmic curves

1 Motivation

Exponential and logarithmic functions arise naturally in mathematics and the sciences. Unlike polynomial and trigonometric functions, these functions are defined using limits and inverse relationships. Their derivatives exhibit remarkably simple and useful forms.

2 The Natural Exponential Function

Definition 2.1. *The* natural exponential function *is the function*

$$f(x) = e^x,$$

where e is the unique real number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

This defining limit determines the derivative of e^x.

Theorem 2.1.

$$\frac{d}{dx}(e^x) = e^x.$$

Proof. Using the limit definition of the derivative,

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}.$$

Factor out e^x:

$$= e^x \lim_{h\to 0} \frac{e^h - 1}{h}.$$

By the defining property of e,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1,$$

so

$$\frac{d}{dx}(e^x) = e^x.$$

□

3 Derivatives of Exponential Functions

Theorem 3.1. *If $f(x) = e^{g(x)}$, then*

$$f'(x) = e^{g(x)} g'(x).$$

Proof. This follows immediately from the chain rule. □

Example 3.1. *Differentiate $f(x) = e^{3x^2}$.*

Let $u = 3x^2$. Then

$$f'(x) = e^u \cdot 6x = 6xe^{3x^2}.$$

Example 3.2. *Differentiate $g(x) = e^{-x}\sin(x)$.*

Using the product rule and chain rule,

$$g'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x).$$

Example 3.3. *Find the equation of the tangent line to $y = e^{x^2}$ at $x = 1$.*

First compute the derivative using the chain rule:

$$\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot 2x.$$

Thus the slope at $x = 1$ is

$$m = 2 \cdot 1 \cdot e^1 = 2e.$$

The point on the curve is $(1, e)$. Therefore, the tangent line is

$$y - e = 2e(x - 1).$$

A sketch of the curve and its tangent line is shown in Figure 1.

4 Exponential Functions with Other Bases

Theorem 4.1. *If $a > 0$ and $a \neq 1$, then*

$$\frac{d}{dx}(a^x) = a^x \ln(a).$$

Remark 4.1. *This formula follows from writing $a^x = e^{x\ln(a)}$ and applying the chain rule.*

Example 4.1. *Differentiate $h(x) = 2^x$.*

$$h'(x) = 2^x \ln(2).$$

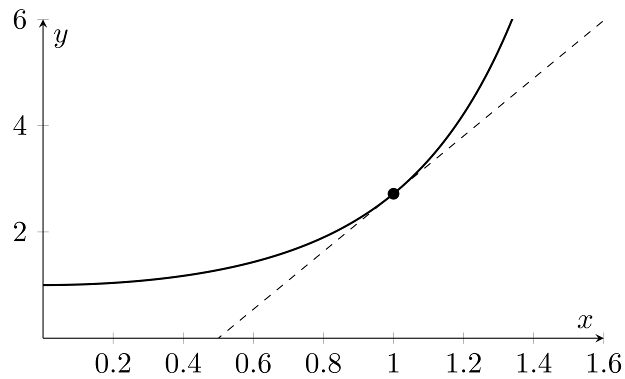

Figure 1: The graph of $y = e^{x^2}$ and the tangent line at $x = 1$.

5 The Natural Logarithm

Definition 5.1. *The* natural logarithm $\ln(x)$ *is the inverse function of* e^x*, defined for* $x > 0$*.*

Theorem 5.1.

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \qquad x > 0.$$

Proof. Since $\ln(x)$ is the inverse of e^x, we have

$$e^{\ln(x)} = x.$$

Differentiate both sides implicitly:

$$e^{\ln(x)} \frac{d}{dx}(\ln(x)) = 1.$$

Substitute $e^{\ln(x)} = x$:

$$x \frac{d}{dx}(\ln(x)) = 1,$$

so

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

□

6 Logarithmic Examples

Example 6.1. *Differentiate* $f(x) = \ln(x^2 + 1)$*.*

Using the chain rule,

$$f'(x) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

Example 6.2. *Differentiate* $g(x) = x\ln(x)$*.*

Using the product rule,

$$g'(x) = \ln(x) + 1.$$

7 Tangent Line Example

Example 7.1. *Find the equation of the tangent line to $y = \ln(x)$ at $x = 1$.*

$$y' = \frac{1}{x}, \qquad y'(1) = 1.$$

The point is $(1, 0)$. The tangent line is

$$y = x - 1.$$

Figure 2 shows the curve and the tangent line at $x = 1$.

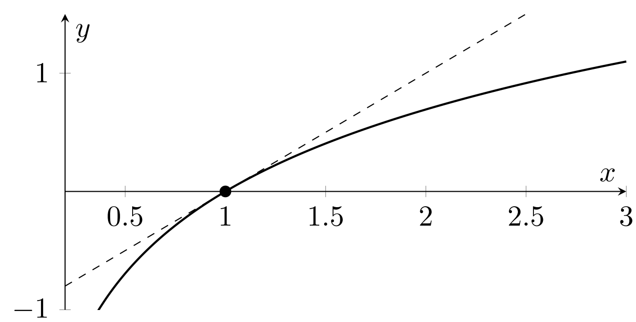

Figure 2: The graph of $y = \ln(x)$ and the tangent line at $x = 1$.

8 Why This Matters for Calculus

Exponential and logarithmic derivatives complete our toolkit for differentiating a wide range of functions.

- They allow us to model rapid growth and decay.
- Their simple derivative rules make them especially powerful.
- They interact naturally with the chain rule and implicit differentiation.
- They are essential for optimization, related rates, and differential equations.