Exponential and Logarithmic Functions

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Math 140: Calculus with Analytic Geometry

1 Learning Goals

After studying these notes, you should be able to:

- recognize exponential growth and decay and describe graph transformations;
- explain why the base e is natural in calculus and interpret exponential models;
- move between logarithmic and exponential form using the inverse relationship;
- derive and apply the product, quotient, and power properties of logarithms;
- solve exponential and logarithmic equations while respecting domain restrictions; and
- connect these algebraic ideas to derivatives used later in Calculus I.

2 Exponential Functions

Polynomial functions have a variable base and a fixed exponent, such as x^3. An exponential function reverses these roles: its base is fixed and its exponent varies.

Definition 2.1. For a fixed number $b > 0$ with $b \neq 1$, the *exponential function with base b* is

$$f(x) = b^x.$$

The restrictions on b ensure that b^x is real and positive for every real x, and that the function is one-to-one.

The familiar exponent laws extend from rational exponents to real exponents:

$$b^{x+y} = b^x b^y, \qquad b^{x-y} = \frac{b^x}{b^y}, \qquad (b^x)^y = b^{xy}, \qquad b^0 = 1, \qquad b^{-x} = \frac{1}{b^x}.$$

Every exponential graph passes through $(0, 1)$ and remains above the x-axis. Thus

$$\mathrm{dom}(b^x) = \mathbb{R}, \qquad \mathrm{range}(b^x) = (0, \infty), \qquad y = 0 \text{ is a horizontal asymptote.}$$

If $b > 1$, then b^x is increasing (growth); if $0 < b < 1$, then b^x is decreasing (decay).

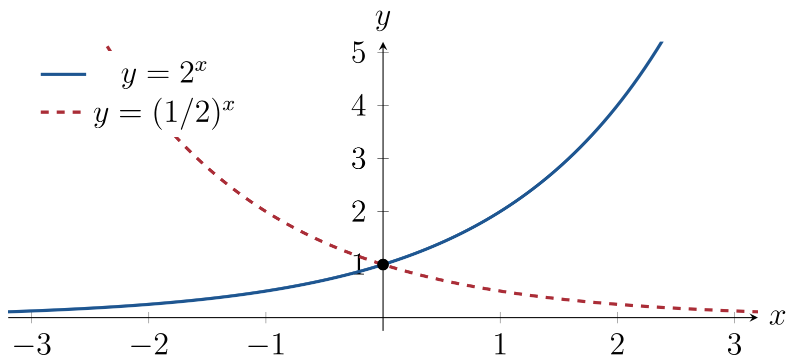

Figure 1: Exponential growth and decay. Both graphs have horizontal asymptote $y = 0$.

2.1 Transformations

For

$$g(x) = Ab^{x-h} + k,$$

h shifts the graph horizontally, k shifts it vertically, and A scales the output (and reflects the graph across the x-axis when $A < 0$). The horizontal asymptote becomes $y = k$.

Example 2.1. For $g(x) = -2 \cdot 3^{x+1} + 4$, the graph of 3^x is shifted left 1, stretched vertically by 2, reflected across the x-axis, and shifted up 4. Its horizontal asymptote is $y = 4$, its domain is $\mathbb{R}$, and its range is $(-\infty, 4)$.

3 Euler's Number and Exponential Models

Among all bases, the number

$$e \approx 2.718281828$$

is especially natural in calculus. It may be introduced through continuous compounding,

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

and later we will see that e^x is the unique exponential function whose slope at $x = 0$ is 1.

Any exponential function can be written using base e:

$$b^x = e^{x \ln b}.$$

An exponential model is commonly written in either form

$$Q(t) = Q_0 b^t \qquad \text{or} \qquad Q(t) = Q_0 e^{kt}.$$

Here $Q_0 = Q(0)$ is the initial amount. In the second form, $k > 0$ describes growth and $k < 0$ describes decay. Equal time intervals produce equal *multiplicative* changes; this is the defining signal of exponential behavior.

Example 3.1. If $Q(t) = Q_0 e^{kt}$, then its doubling time T_d satisfies

$$2Q_0 = Q_0 e^{kT_d} \quad \Longrightarrow \quad T_d = \frac{\ln 2}{k} \quad (k > 0).$$

For decay, its half-life is $T_{1/2} = \ln(2)/|k|$. Notice that the initial amount cancels: doubling time and half-life depend only on the rate constant.

4 Logarithms as Inverse Functions

Because b^x is one-to-one, it has an inverse.

Definition 4.1. For $b > 0$ and $b \neq 1$, the *logarithm with base* b is defined by

$$\log_b(x) = y \quad \Longleftrightarrow \quad b^y = x.$$

The notation $\ln x$ means $\log_e x$; it is read "the natural logarithm of x."

The domain and range swap under inversion:

$$\operatorname{dom}(\log_b x) = (0, \infty), \qquad \operatorname{range}(\log_b x) = \mathbb{R}, \qquad x = 0 \text{ is a vertical asymptote.}$$

The inverse identities are

$$\log_b(b^x) = x \quad (x \in \mathbb{R}), \qquad b^{\log_b x} = x \quad (x > 0).$$

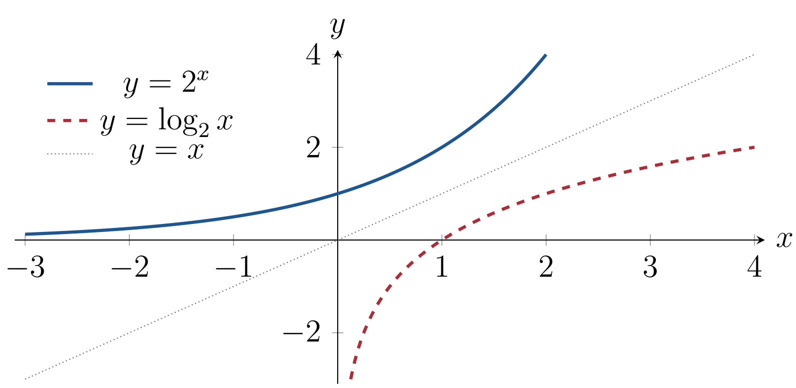

Figure 2: An exponential function and its inverse are reflections across $y = x$.

Example 4.1.

$$\log_2 8 = 3 \iff 2^3 = 8, \qquad \log_{1/2}\left(\frac{1}{4}\right) = 2 \iff \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Also, $\ln(e^{-4}) = -4$ and $e^{\ln 7} = 7$.

4.1 Properties of Logarithms

The logarithm properties are not separate rules to memorize: they are the exponent laws written in the language of inverse functions. Let

$$u = \log_b M \qquad \text{and} \qquad v = \log_b N,$$

where $M > 0$ and $N > 0$. By the definition of a logarithm,

$$M = b^u \qquad \text{and} \qquad N = b^v.$$

For the product property, the product rule for exponents gives

$$MN = b^u b^v = b^{u+v}.$$

Taking the base-b logarithm undoes the exponential:

$$\log_b(MN) = \log_b\left(b^{u+v}\right) = u + v = \log_b M + \log_b N.$$

Similarly, the quotient rule for exponents gives

$$\frac{M}{N} = \frac{b^u}{b^v} = b^{u-v},$$

and therefore

$$\log_b\left(\frac{M}{N}\right) = u - v = \log_b M - \log_b N.$$

Finally, for a real number r for which the expression is defined,

$$M^r = (b^u)^r = b^{ru},$$

so

$$\log_b(M^r) = ru = r\log_b M.$$

Thus, for $M > 0$, $N > 0$, and real r, the three principal properties are

$$\begin{aligned} \log_b(MN) &= \log_b M + \log_b N, \\ \log_b\left(\frac{M}{N}\right) &= \log_b M - \log_b N, \qquad \log_b 1 = 0, \quad \log_b b = 1. \\ \log_b(M^r) &= r\log_b M. \end{aligned}$$

The last two identities follow immediately from $b^0 = 1$ and $b^1 = b$.

The change-of-base formula lets us evaluate any base using natural logarithms:

$$\log_b x = \frac{\ln x}{\ln b}.$$

Indeed, if $y = \log_b x$, then $b^y = x$. Taking natural logarithms gives $y \ln b = \ln x$.

Example 4.2. Assuming $x > 0$, expand

$$\ln\left(\frac{x^3\sqrt{x+1}}{e^{2x}}\right).$$

Every logarithm below has a positive argument, and the logarithm laws give

$$3\ln x + \frac{1}{2}\ln(x+1) - \ln(e^{2x}) = 3\ln x + \frac{1}{2}\ln(x+1) - 2x.$$

Example 4.3. For positive x and y,

$$2\ln x - \frac{1}{3}\ln y + \ln 5 = \ln(x^2) - \ln(y^{1/3}) + \ln 5 = \ln\left(\frac{5x^2}{\sqrt[3]{y}}\right).$$

4.2 Solving Exponential Equations

First isolate the exponential expression. If both sides can be written with the same base, equate exponents. Otherwise, take a logarithm of both sides.

Example 4.4. Solve $4^{x-1} = 8$. Since $4 = 2^2$ and $8 = 2^3$,

$$2^{2x-2} = 2^3 \quad \Longrightarrow \quad 2x - 2 = 3 \quad \Longrightarrow \quad x = \frac{5}{2}.$$

Example 4.5. Solve $3^{2x-1} = 7$. Taking natural logarithms of both positive sides gives

$$(2x-1)\ln 3 = \ln 7.$$

Therefore

$$x = \frac{1}{2}\left(1 + \frac{\ln 7}{\ln 3}\right).$$

Substitution verifies the solution exactly.

4.3 Solving Logarithmic Equations

State the domain restrictions *before* using logarithm laws. Then rewrite in exponential form or combine logarithms. Finally, check proposed answers in the original equation; algebra can introduce values where a logarithm is undefined.

Example 4.6. Solve

$$\ln(x-1)+\ln(x+3)=\ln(5x+7).$$

The three logarithms require

$$x-1>0, \qquad x+3>0, \qquad 5x+7>0,$$

so the common domain is $x>1$. Within this domain,

$$\ln\Big((x-1)(x+3)\Big)=\ln(5x+7).$$

Since ln is one-to-one,

$$(x-1)(x+3)=5x+7 \Longrightarrow x^2-3x-10=0 \Longrightarrow (x-5)(x+2)=0.$$

Only $x=5$ lies in the original domain; $x=-2$ is extraneous.

Example 4.7. Solve $\log_2(x-3)=4$. Rewriting in exponential form,

$$x-3=2^4=16,$$

so $x=19$, which satisfies $x-3>0$.

5 Why This Matters for Calculus

Exponential and logarithmic functions occur throughout Calculus I and II. The algebra in these notes supports several central ideas:

- **Limits and continuity.** Exponential functions are continuous on $\mathbb{R}$, and logarithmic functions are continuous on $(0,\infty)$. Consequently, many limits involving these functions can be evaluated by direct substitution whenever the input remains in the domain.
- **Derivative rules.** The base e is chosen so that the natural exponential function is its own derivative. Since $\ln x$ is its inverse,

  $$\frac{d}{dx}e^x=e^x, \qquad \frac{d}{dx}\ln x=\frac{1}{x} \quad (x>0).$$

  The change-of-base formulas then give

  $$\frac{d}{dx}b^x=b^x\ln b, \qquad \frac{d}{dx}\log_b x=\frac{1}{x\ln b}.$$

- **The chain rule.** Exponential and logarithmic expressions usually contain an inner function. For example, differentiating e^{x^2} or $\ln(3x+1)$ requires multiplying by the derivative of the exponent or logarithm argument.

- **Logarithmic differentiation.** Because logarithms turn products into sums, quotients into differences, and powers into coefficients, they simplify derivatives of complicated products and variable powers such as x^x.

- **Growth and decay models.** Population growth, radioactive decay, continuous compounding, and Newton's law of cooling lead to exponential models. Later, the differential equation

$$Q'(t) = kQ(t)$$

  will characterize quantities whose instantaneous rate of change is proportional to their current amount; its solutions have the form $Q(t) = Q_0 e^{kt}$.

- **Antiderivatives and integration.** Since the derivative of $\ln x$ is $1/x$, logarithms appear when integrating reciprocal expressions. Likewise, e^x is its own antiderivative. These facts become basic tools in substitution, differential equations, and improper integrals.