The First Derivative Test and Classifying Extrema

Math 140: Calculus with Analytic Geometry

Key Topics

- Critical numbers
- Local extrema
- Why extrema must be critical points
- The First Derivative Test
- Absolute extrema on closed intervals

1 Critical Numbers

Definition 1.1

Let f be defined on an interval I. A number $c \in I$ is called a **critical number** if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Critical numbers are candidates for local extrema.

Important: Not every critical number corresponds to a local maximum or minimum.

2 Local Extrema

Definition 2.1

A function f has a:

- **Local maximum** at c if $f(c) \geq f(x)$ for all x near c.
- **Local minimum** at c if $f(c) \leq f(x)$ for all x near c.

Why Local Extrema Must Be Critical Points

Suppose f has a local maximum at c and is differentiable at c.

Then for x near c,

$$f(x) \leq f(c).$$

For $x > c$ sufficiently close to c,

$$\frac{f(x) - f(c)}{x - c} \leq 0.$$

For $x < c$ sufficiently close to c,

$$\frac{f(x) - f(c)}{x - c} \geq 0.$$

If the derivative exists, the left-hand and right-hand limits must agree. The only possibility is

$$f'(c) = 0.$$

The same reasoning applies to local minima.

Conclusion:

If f has a local extremum at c and f is differentiable at c, then c is a critical number.

3 The First Derivative Test

Theorem 3.1 (First Derivative Test)

Let c be a critical number of f, and suppose f is continuous near c.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' does not change sign at c, then f has no local extremum at c.

Why the First Derivative Test Works

If $f'(x) > 0$ on an interval, then f is increasing there.

If $f'(x) < 0$ on an interval, then f is decreasing there.

This follows directly from the Mean Value Theorem:

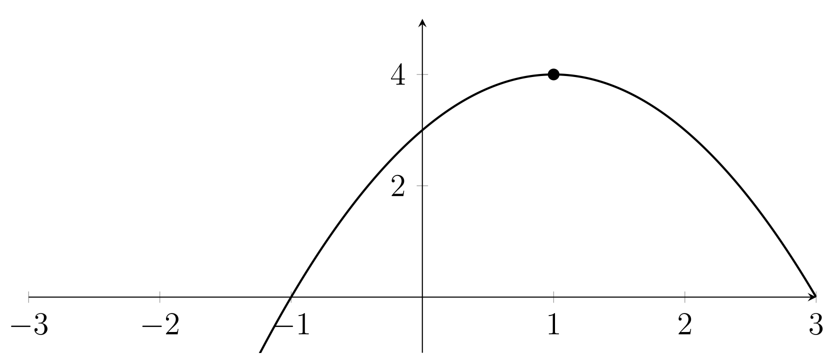
For $x_1 < x_2$,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2.

Thus the sign of f' determines monotonicity.

Illustration



Here $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$, so $x = 1$ is a local maximum.

4 Examples

Example 4.1

Classify the critical points of

$$f(x) = x^3 - 3x^2 + 2.$$

Solution.

$$f'(x) = 3x^2 - 6x = 3x(x - 2).$$

Critical numbers:

$$x = 0, \quad x = 2.$$

Sign chart:

x	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$+$	$-$	$+$

Therefore:
- At $x = 0$, f' changes from positive to negative $\rightarrow$ local maximum. - At $x = 2$, f' changes from negative to positive $\rightarrow$ local minimum.

Example 4.2 (No Extremum)

$$f(x) = x^3.$$

$$f'(x) = 3x^2.$$

Critical number: $x = 0$.
But $f'(x) \geq 0$ on both sides of 0.
Thus, $x = 0$ is not a local extremum.

5 Absolute Extrema on Closed Intervals

Theorem 5.1 (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$.

Procedure

To find absolute extrema on $[a, b]$:

1. Find critical numbers in (a, b).
2. Evaluate f at each critical number.
3. Evaluate $f(a)$ and $f(b)$.
4. Compare all values.

Example 5.1

Find the absolute extrema of

$$f(x) = x^3 - 3x^2 + 2 \quad \text{on } [-1, 3].$$

Critical numbers: 0 and 2.
Evaluate:

$$f(-1) = -2, \quad f(0) = 2, \quad f(2) = -2, \quad f(3) = 2.$$

Absolute maximum: 2 Absolute minimum: -2

6 Why This Matters

The First Derivative Test:

- Classifies critical points using sign changes.
- Connects derivative sign to increasing/decreasing behavior.
- Provides the foundation for optimization problems.
- Bridges local derivative behavior and global function structure.

This is one of the central tools of differential calculus. It allows us to understand the global shape of a function using only its derivative.