The Fundamental Theorem of Calculus

Math 140: Calculus with Analytic Geometry

1 Introduction

In the previous lecture, we evaluated the integral

$$\int_a^b e^x \, dx$$

by choosing sample points in a clever way so that the Riemann sum telescoped and produced

$$e^b - e^a.$$

This suggests a deeper connection between definite integrals and antiderivatives. In particular, if $F'(x) = f(x)$, then it appears that

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

The goal of this lecture is to justify this relationship using the Mean Value Theorem.

2 Review of the Mean Value Theorem

We begin by recalling the Mean Value Theorem.

Theorem (Mean Value Theorem)

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example

Let $f(x) = x^2$ on $[1, 3]$. Then

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Since $f'(x) = 2x$, we solve

$$2c = 4 \quad \Rightarrow \quad c = 2.$$

Thus, the Mean Value Theorem guarantees a point $c = 2$ where the instantaneous rate of change equals the average rate of change.

3 Applying the Mean Value Theorem on Subintervals

Let F be a function such that $F'(x) = f(x)$. We divide $[a, b]$ into subintervals of equal width Δx and define

$$x_i = a + i\Delta x.$$

On each subinterval $[x_{i-1}, x_i]$, the Mean Value Theorem guarantees a point $c_i \in (x_{i-1}, x_i)$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

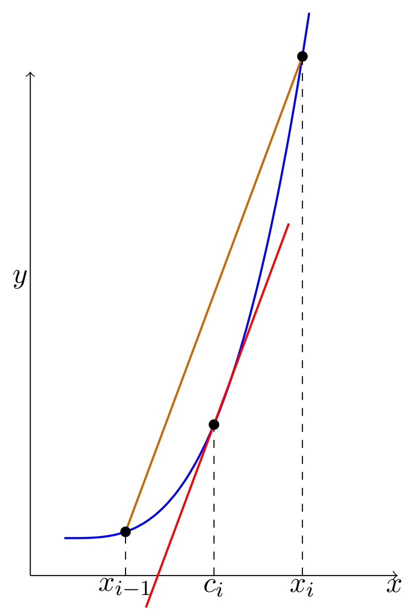
Since $F'(c_i) = f(c_i)$, this becomes

$$f(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Multiplying by Δx, we obtain

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

Figure



The point c_i is chosen so that the slope of the tangent line at c_i equals the slope of the secant line across the interval.

4 The Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$, and suppose F is an antiderivative of f, that is, $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

Divide $[a, b]$ into n subintervals of equal width Δx. Then, as shown above, for each subinterval there exists $c_i \in [x_{i-1}, x_i]$ such that

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

Summing over $i = 1$ to n, we obtain

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big).$$

The right-hand side is a telescoping sum:

$$\sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) = F(x_n) - F(x_0) = F(b) - F(a).$$

Taking the limit as $n \to \infty$, the left-hand side becomes the definite integral:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

5 Examples

Example

Evaluate

$$\int_0^2 x^2\,dx.$$

An antiderivative is $F(x) = \frac{x^3}{3}$. Thus,

$$\int_0^2 x^2\,dx = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

Example

Evaluate

$$\int_0^1 e^x\,dx.$$

An antiderivative is $F(x) = e^x$. Thus,

$$\int_0^1 e^x\,dx = e - 1.$$

6 Summary

The Fundamental Theorem of Calculus shows that definite integrals and antiderivatives are fundamentally connected. While Riemann sums define the integral as a limit of area approximations, the theorem shows that we can compute the exact value using antiderivatives:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This result allows us to evaluate definite integrals efficiently without computing limits of sums.