Fundamental Theorem of Calculus

Math 140: Calculus with Analytic Geometry

1 The Fundamental Theorem of Calculus

In *Riemann Sums*, we chose sample points so that neighboring terms canceled. The Mean Value Theorem explains why such a choice is possible whenever we have an antiderivative. The resulting formula gives the exact value of a definite integral from two endpoint values.

Theorem 1.1 (Fundamental Theorem of Calculus, evaluation formula). Let $a < b$ and let f be continuous on $[a, b]$. If F is continuous on $[a, b]$ and $F'(x) = f(x)$ for $a < x < b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. For each positive integer n, divide $[a, b]$ into n equal subintervals with endpoints $a = x_0 < x_1 < \cdots < x_n = b$ and width $\Delta x = (b - a)/n$. On $[x_{i-1}, x_i]$, F is continuous and differentiable on the interior. The Mean Value Theorem gives a point c_i in (x_{i-1}, x_i) such that

$$f(c_i) = F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Multiplying by Δx gives the signed contribution $f(c_i)\,\Delta x$ as a difference of endpoint values. Adding gives

$$\begin{aligned}\sum_{i=1}^{n} f(c_i)\,\Delta x &= \sum_{i=1}^{n}\Big(F(x_i) - F(x_{i-1})\Big)\\ &= \Big(F(x_1) - F(x_0)\Big) + \cdots + \Big(F(x_n) - F(x_{n-1})\Big)\\ &= F(b) - F(a).\end{aligned}$$

Every interior endpoint value occurs once with each sign. Since f is continuous, the Riemann sums converge to the definite integral for any choice of sample points, including these choices from the Mean Value Theorem. Thus

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = F(b) - F(a). \qquad \square$$

Definition 1.1 (Evaluation notation). The vertical bar in $F(x)\big|_a^b$ means to evaluate F at the upper limit b and subtract its value at the lower limit a. Thus

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

Any antiderivative works. If we use $F(x) + C$, the constant cancels because $(F(b) + C) - (F(a) + C) = F(b) - F(a)$. In the examples, we choose $C = 0$.

2 Comparison with Geometry

Example 2.1 (A rectangle). Evaluate $\int_0^3 3\,dx$.

Since $\dfrac{d}{dx}(3x) = 3$, we have

$$\int_0^3 3\,dx = 3x\Big|_0^3 = 3(3) - 3(0) = 9.$$

The shaded rectangle has width 3 and height 3, so its area is also 9. This is the same integral we evaluated by telescoping a Riemann sum.

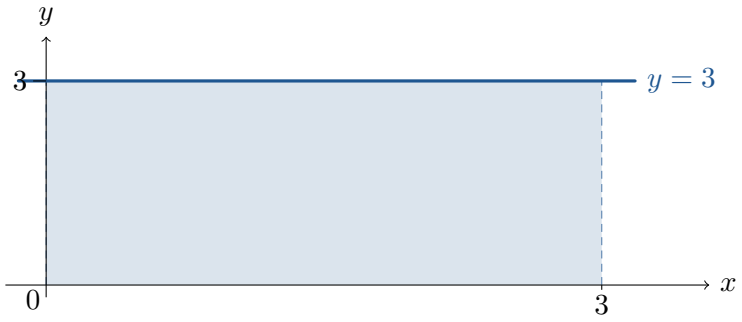


Figure 1: The region is a rectangle of width 3 and height 3.

Example 2.2 (A trapezoid). Evaluate $\int_1^3 2x\,dx$.

An antiderivative of $2x$ is x^2, so

$$\int_1^3 2x\,dx = x^2\Big|_1^3 = 3^2 - 1^2 = 8.$$

The trapezoid has vertical sides of lengths 2 and 6 separated by $3 - 1 = 2$ units. Its area is $\frac{1}{2}(2+6)\cdot 2 = 8$.

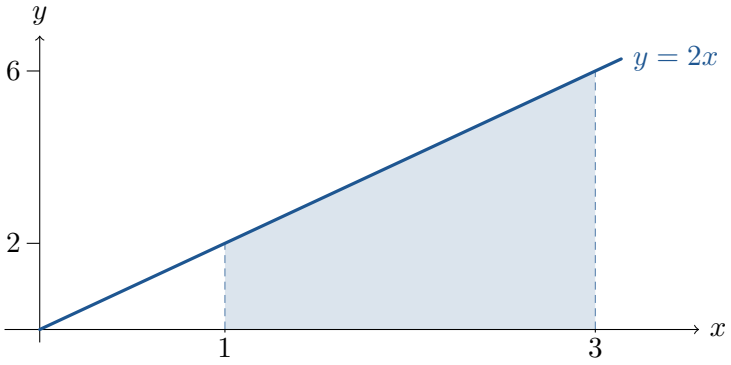


Figure 2: The parallel sides of the trapezoid have lengths 2 and 6.

3 Powers and Radicals

Example 3.1 (The quadratic region from Riemann sums). Evaluate $\displaystyle\int_0^2 x^2\,dx$.

Differentiating x^3 gives $3x^2$, so the factor $1/3$ makes $F(x) = x^3/3$ an antiderivative of x^2. The FTC gives

$$\int_0^2 x^2\,dx = \frac{x^3}{3}\bigg|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

The factor $1/3$ is the same factor that appeared in each difference of cubes in the telescoping Riemann sum.

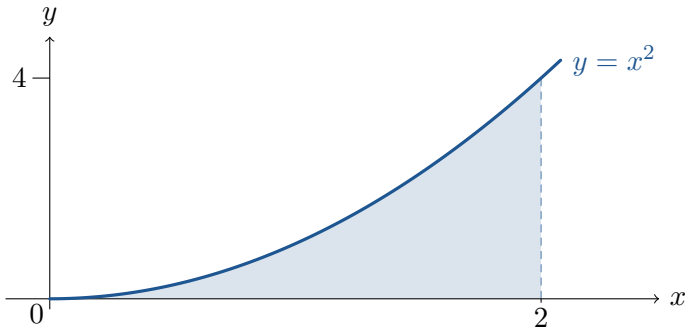


Figure 3: The area under $y = x^2$ on $[0, 2]$ is $8/3$.

Example 3.2 (A fractional power). Evaluate $\displaystyle\int_1^4 \sqrt{x}\,dx$.

Write $\sqrt{x} = x^{1/2}$. The power rule for antiderivatives gives $F(x) = \frac{2}{3}x^{3/2}$. At each endpoint, use $x^{3/2} = (\sqrt{x})^3$.

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg|_1^4 = \frac{2}{3}\left(2^3 - 1^3\right) = \frac{14}{3}.$$

The graph rises from height 1 to height 2 over an interval of width 3, so the area lies between 3 and 6.

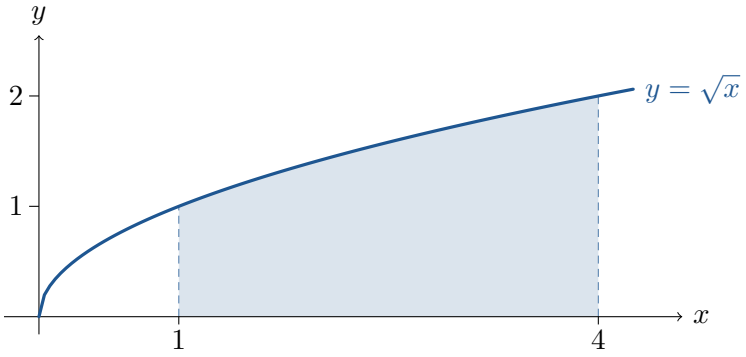


Figure 4: The curve rises from height 1 to height 2.

4 Exponential and Reciprocal Functions

Example 4.1 (An exponential with a linear exponent). Evaluate $\displaystyle\int_0^1 e^{2x}\,dx$.

The chain rule gives $\dfrac{d}{dx}e^{2x} = 2e^{2x}$. Dividing by 2 gives an antiderivative of the integrand.

$$\int_0^1 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_0^1 = \frac{1}{2}e^2 - \frac{1}{2}e^0 = \frac{e^2-1}{2}.$$

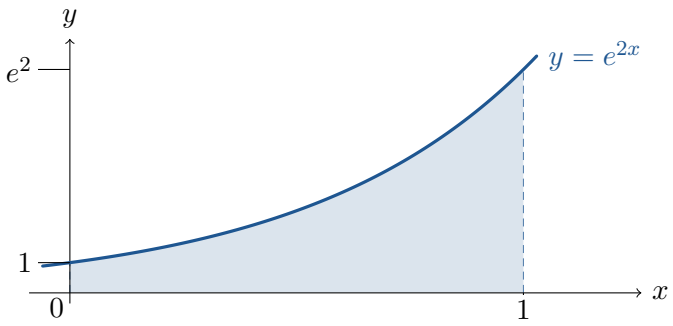


Figure 5: The exponential region extends from $x = 0$ to $x = 1$.

Example 4.2 (A reciprocal and its logarithmic antiderivative). Evaluate $\displaystyle\int_1^4 \frac{1}{x}\,dx$.

The interval lies in $x > 0$, where $\dfrac{d}{dx}\ln x = 1/x$. Therefore

$$\int_1^4 \frac{1}{x}\,dx = \ln x\big|_1^4 = \ln 4 - \ln 1 = \ln 4.$$

On an interval of negative x-values, we would use $\ln|x|$. Before applying the FTC, check continuity across the entire interval. The reciprocal is undefined at 0, so this theorem does not apply on an interval that contains 0.

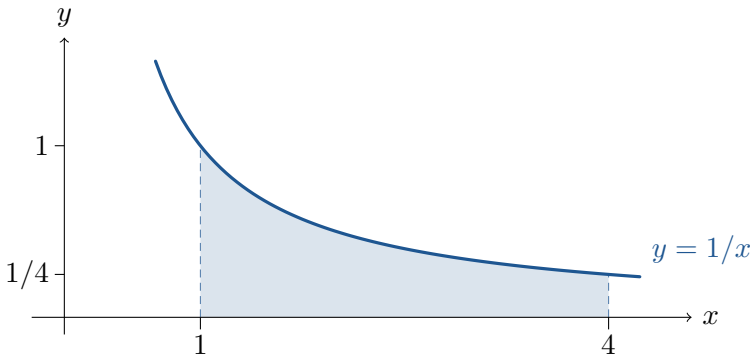


Figure 6: The reciprocal decreases from 1 to 1/4 on $[1, 4]$.

5 Trigonometric Functions and Signed Area

Example 5.1 (A sine integral). Evaluate $\displaystyle\int_0^{\pi/2} \sin x\,dx$.

Since $\dfrac{d}{dx}(-\cos x) = \sin x$, we obtain

$$\int_0^{\pi/2} \sin x\,dx = -\cos x\Big|_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) - \Big(-\cos 0\Big) = 1.$$

The subtraction includes the entire lower endpoint value, which is -1.

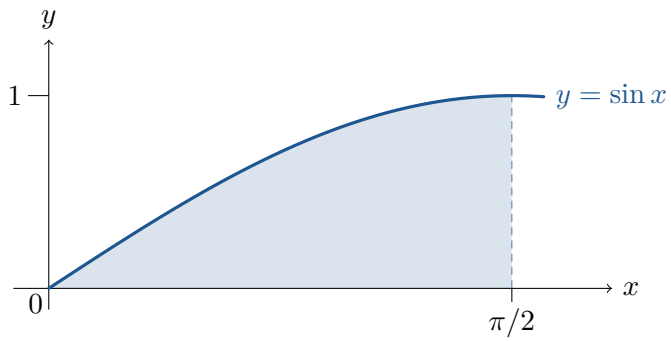


Figure 7: The sine curve is above the axis between 0 and $\pi/2$.

Example 5.2 (A cosine integral with both signs). Evaluate $\displaystyle\int_0^{\pi} \cos x\,dx$.

The cosine is continuous on the whole interval, so one endpoint evaluation gives the signed integral.

$$\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0.$$

To see the cancellation, use the FTC on each side of $\pi/2$. The contribution on $[0, \pi/2]$ is $1 - 0 = 1$, and the contribution on $[\pi/2, \pi]$ is $0 - 1 = -1$. The total geometric area is $1 + 1 = 2$.

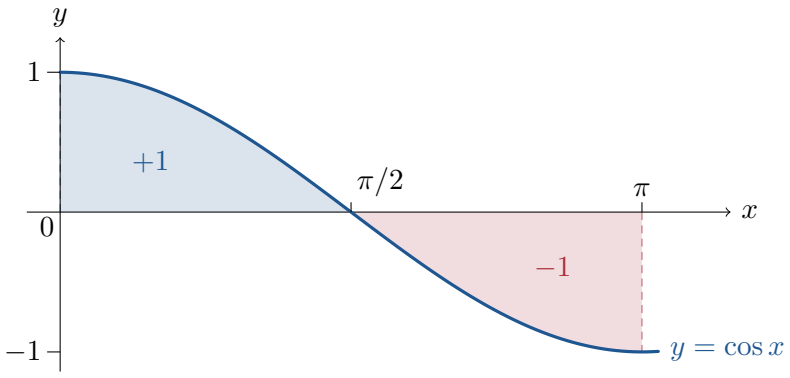


Figure 8: The two shaded regions have signed contributions 1 and -1.

6 Finding a Suitable Antiderivative

Example 6.1 (A logarithmic integrand). Evaluate $\displaystyle\int_1^e \ln x\,dx$.

The product rule gives $(x\ln x)' = \ln x + 1$ for $x > 0$. Subtracting x removes the extra 1 from the derivative, so $F(x) = x\ln x - x$ is an antiderivative of $\ln x$. Thus

$$\int_1^e \ln x\,dx = x\ln x - x\Big|_1^e = (e\cdot 1 - e) - (1\cdot 0 - 1) = 1.$$

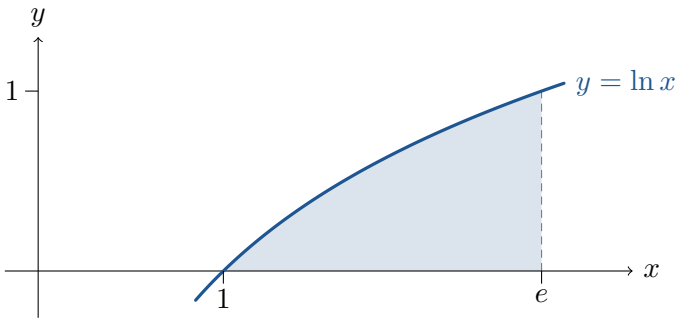


Figure 9: The logarithmic region has area 1.

Example 6.2 (An absolute value). Evaluate $\displaystyle\int_0^2 |x^2 - 1|\,dx$.

The expression $x^2 - 1$ changes sign at $x = 1$ on $[0, 2]$. On $[0, 1]$, the integrand is $1 - x^2$. On $[1, 2]$, it is $x^2 - 1$. Split at 1 and apply the FTC on each interval.

$$\begin{aligned}\int_0^2 |x^2-1|\,dx &= \int_0^1 (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx \\ &= x - \frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3} - x\Big|_1^2 \\ &= \frac{2}{3} + \left(\frac{2}{3} - \left(-\frac{2}{3}\right)\right) = 2.\end{aligned}$$

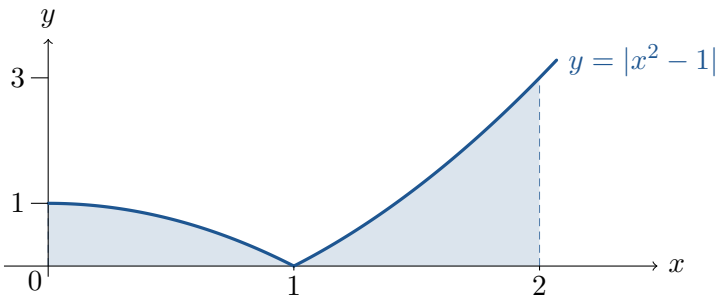


Figure 10: Both shaded regions contribute positively to the integral.