Higher-Order Derivatives

Math 140: Calculus with Analytic Geometry

Key Topics

- Definition and notation for higher-order derivatives
- Computing second and higher derivatives
- Repeated application of derivative rules
- Higher-order derivatives of polynomial and trigonometric functions
- Tangent lines and horizontal tangents
- Using the second derivative to describe concavity

1 Motivation

Up to this point, we have focused on finding the first derivative of a function. In many situations, it is useful to differentiate a function more than once. This leads to the concept of higher-order derivatives.

2 Definition of Higher-Order Derivatives

Definition 2.1. *Let f be a function whose derivative exists.*

- *The* first derivative *of f is denoted by $f'(x)$.*
- *If $f'(x)$ is differentiable, the* second derivative *of f is defined by*

$$f''(x) = \frac{d}{dx}\big(f'(x)\big).$$

- *More generally, if the $(n-1)$st derivative of f exists and is differentiable, then the n*th derivative *of f is defined recursively by*

$$f^{(n)}(x) = \frac{d}{dx}\big(f^{(n-1)}(x)\big), \qquad n \geq 2.$$

Remark 2.1. *Common notations include $f''(x)$, $f^{(2)}(x)$, and $\dfrac{d^2}{dx^2}f(x)$.*

3 Polynomial Examples

Example 3.1. *Let $f(x) = x^4 - 3x^2 + 2$. Compute $f'(x)$ and $f''(x)$.*

$$f'(x) = 4x^3 - 6x, \qquad f''(x) = 12x^2 - 6.$$

Example 3.2. *Find all points where the tangent line to $f(x) = x^4 - 3x^2 + 2$ is horizontal.*

Horizontal tangents occur where $f'(x) = 0$:

$$4x^3 - 6x = 2x(2x^2 - 3) = 0,$$

so $x = 0$ and $x = \pm\sqrt{\frac{3}{2}}$.

4 Trigonometric Examples

Example 4.1. *Let $g(x) = \sin(x)$. Compute the first four derivatives.*

$$\begin{aligned}
g'(x) &= \cos(x), \\
g''(x) &= -\sin(x), \\
g^{(3)}(x) &= -\cos(x), \\
g^{(4)}(x) &= \sin(x).
\end{aligned}$$

Remark 4.1. *The derivatives of $\sin(x)$ and $\cos(x)$ repeat every four derivatives.*

Example 4.2. *Differentiate $h(x) = x\cos(x)$ twice.*

Using the product rule,

$$h'(x) = \cos(x) - x\sin(x),$$

and

$$h''(x) = -\sin(x) - \sin(x) - x\cos(x) = -2\sin(x) - x\cos(x).$$

5 Rational and Mixed Examples

Example 5.1. *Let $p(x) = \dfrac{\sin(x)}{x}$. Find $p'(x)$.*

Using the quotient rule,

$$p'(x) = \frac{x\cos(x) - \sin(x)}{x^2}.$$

Example 5.2. *Differentiate $q(x) = x^2\sin(x)$ twice.*

$$\begin{aligned}
q'(x) &= 2x\sin(x) + x^2\cos(x), \\
q''(x) &= 2\sin(x) + 4x\cos(x) - x^2\sin(x).
\end{aligned}$$

6 Tangent Line Example

Example 6.1. *Find the equation of the tangent line to* $y = \cos(x)$ *at* $x = \frac{\pi}{4}$.

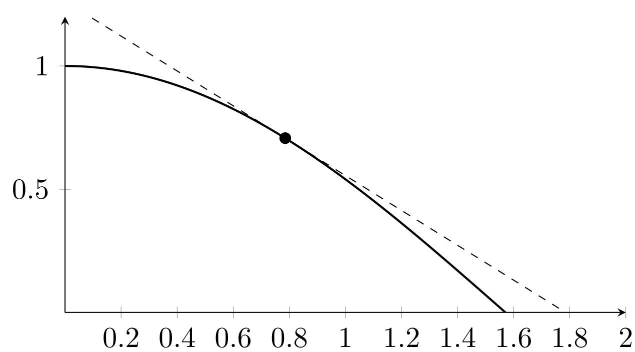
$$y' = -\sin(x), \qquad y'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

The point on the curve is

$$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right).$$

Thus, the tangent line is

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right).$$



7 Concavity and the Second Derivative

In addition to describing the slope of a function, derivatives also provide information about the *shape* of a graph.

Definition 7.1. *Let* f *be a twice-differentiable function.*

- *The graph of* f *is* concave up *on an interval if the slopes of the tangent lines are increasing on that interval.*
- *The graph of* f *is* concave down *on an interval if the slopes of the tangent lines are decreasing on that interval.*

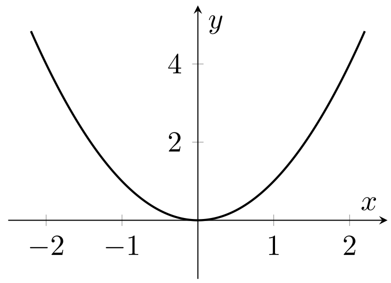
The second derivative provides a convenient test for concavity.

Remark 7.1.
- *If* $f''(x) > 0$ *on an interval, then* f *is concave up on that interval.*
- *If* $f''(x) < 0$ *on an interval, then* f *is concave down on that interval.*

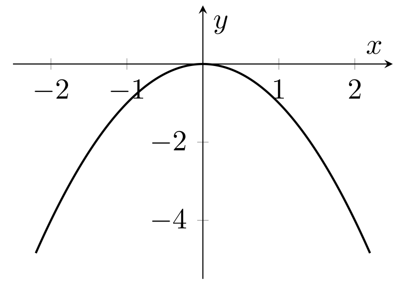
Geometrically, concavity describes the direction in which the curve bends:

- Concave up graphs bend upward, like the graph of $y = x^2$.

$y = x^2$ (concave up)



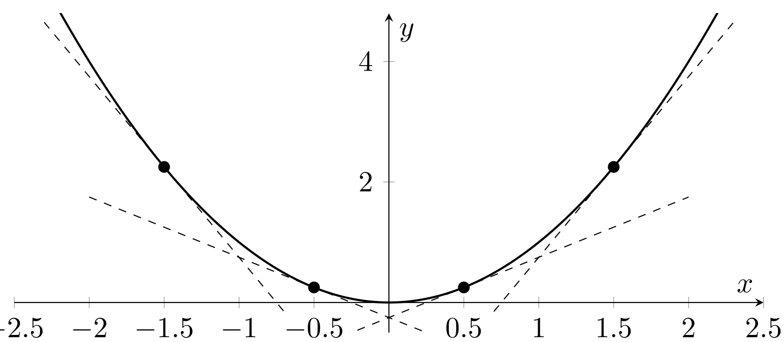
$y = -x^2$ (concave down)



- Concave down graphs bend downward, like the graph of $y = -x^2$.

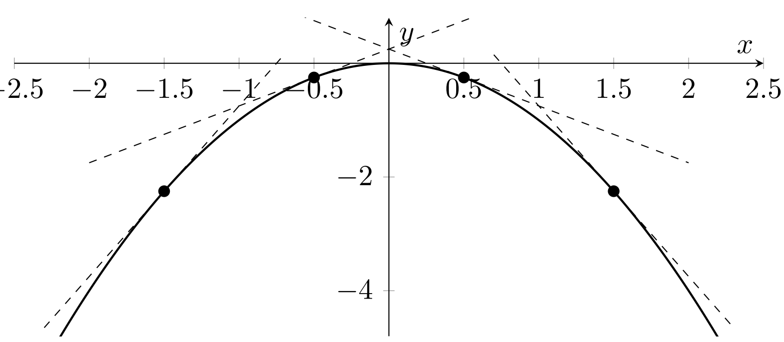
Remark 7.2. *Concavity can be seen by watching how the* slope *changes as you move left to right. On a concave up graph the slopes increase (tangent lines tilt upward more and more). On a concave down graph the slopes decrease (tangent lines tilt downward more and more).*

Tangent slopes increasing on a concave up graph



Remark 7.3. *In the figure above, the tangent line slopes go from negative (left) to positive (right), so the slopes are* increasing. *This is the visual meaning of* concave up.

Tangent slopes decreasing on a concave down graph



Remark 7.4. *In the figure above, the tangent line slopes go from positive (left) to negative (right), so the slopes are* decreasing. *This is the visual meaning of* concave down.

8 Why This Matters for Calculus

Higher-order derivatives allow us to study not only the slope of a function, but how that slope itself changes.

- They provide a systematic way to analyze the shape of graphs.
- They help identify important geometric features such as flat points and changes in curvature.
- They reinforce the use of derivative rules through repeated application.
- They prepare us for curve sketching, optimization, and later applications.