

# Homework 1

## Math 140: Calculus with Analytic Geometry

### Answer Key

#### 1. Polynomial equations.

Solve each equation over the real numbers.

(a)  $2x^3 + x^2 - 8x - 4 = 0$

(b)  $3x^2 - 5x - 4 = 0$

**Solution.** Factoring by grouping gives

$$2x^3 + x^2 - 8x - 4 = x^2(2x + 1) - 4(2x + 1) = (2x + 1)(x - 2)(x + 2).$$

Thus the solutions in part (a) are  $x = -\frac{1}{2}, -2, 2$ . For part (b), the quadratic formula gives

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(-4)}}{6} = \frac{5 \pm \sqrt{73}}{6}.$$

#### 2. Rational expressions and graph features.

Let

$$r(x) = \frac{x^2 - 5x + 6}{x^2 - x - 6}.$$

- (a) Factor and simplify  $r(x)$ , retaining all restrictions from the original expression.
- (b) State the domain of  $r$  in interval notation.
- (c) Identify any holes and all vertical and horizontal asymptotes of the graph.

**Solution.** Since

$$r(x) = \frac{(x - 2)(x - 3)}{(x - 3)(x + 2)} = \frac{x - 2}{x + 2}, \quad x \neq -2, 3,$$

the domain is  $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$ . The canceled factor creates a hole at

$$\left(3, \frac{3 - 2}{3 + 2}\right) = \left(3, \frac{1}{5}\right).$$

The vertical asymptote is  $x = -2$ , and the horizontal asymptote is  $y = 1$ .

#### 3. Composition and domain.

Let

$$f(x) = \frac{1}{x + 1} \quad \text{and} \quad g(x) = \sqrt{4 - x}.$$

Find formulas for  $(f \circ g)(x)$  and  $(g \circ f)(x)$ . State the domain of each composite function in interval notation.

**Solution.** We have

$$(f \circ g)(x) = \frac{1}{\sqrt{4-x}+1}.$$

The square root requires  $x \leq 4$ , and its nonnegative value can never equal  $-1$ . Thus the domain is  $(-\infty, 4]$ . Also,

$$(g \circ f)(x) = \sqrt{4 - \frac{1}{x+1}} = \sqrt{\frac{4x+3}{x+1}}.$$

The radicand must be nonnegative and  $x \neq -1$ . A sign chart with critical numbers  $-1$  and  $-3/4$  gives the domain  $(-\infty, -1) \cup [-3/4, \infty)$ .

#### 4. Inverse functions.

Let  $h(x) = (3x + 5)/(2x - 1)$ .

- (a) State the domain of  $h$ .
- (b) Find  $h^{-1}(x)$  and state its domain.
- (c) Verify your formula by showing that  $h^{-1}(h(x)) = x$ .

**Solution.** The domain of  $h$  is  $\mathbb{R} \setminus \{1/2\}$ . Solving

$$y = \frac{3x+5}{2x-1}$$

for  $x$  gives  $2xy - y = 3x + 5$ , so  $x(2y - 3) = y + 5$ . Therefore

$$h^{-1}(x) = \frac{x+5}{2x-3}, \quad \text{dom}(h^{-1}) = \mathbb{R} \setminus \{3/2\}.$$

Moreover,

$$h^{-1}(h(x)) = \frac{\frac{3x+5}{2x-1} + 5}{2\frac{3x+5}{2x-1} - 3} = \frac{13x/(2x-1)}{13/(2x-1)} = x.$$

#### 5. Exponential functions and models.

- (a) For  $p(x) = -3 \cdot 2^{x-1} + 5$ , describe the transformations of  $y = 2^x$  and state the domain, range, and horizontal asymptote of  $p$ .
- (b) A quantity is modeled by  $Q(t) = 240e^{kt}$  and doubles every 6 years. Find the exact value of  $k$  and then find  $Q(15)$  in exact form.

**Solution.** The graph is shifted right 1, stretched vertically by 3, reflected across the  $x$ -axis, and shifted up 5. Its domain is  $\mathbb{R}$ , its range is  $(-\infty, 5)$ , and its horizontal asymptote is  $y = 5$ . The doubling condition gives  $480 = 240e^{6k}$ , so  $e^{6k} = 2$  and

$$k = \frac{\ln 2}{6}.$$

Consequently,

$$Q(15) = 240e^{15 \ln(2)/6} = 240e^{(5/2) \ln 2} = 240 \cdot 2^{5/2} = 960\sqrt{2}.$$

## 6. Logarithms and exponential equations.

- (a) Condense  $2 \ln(x+1) - \frac{1}{2} \ln x - \ln 3$  into one logarithm, assuming  $x > 0$ .
- (b) Solve  $5^{x+1} = 3^{2x}$ , expressing the answer using logarithms.
- (c) Solve  $\ln(x-2) + \ln x = \ln(3x+4)$ , checking all domain restrictions.

**Solution.** For part (a),

$$2 \ln(x+1) - \frac{1}{2} \ln x - \ln 3 = \ln \left( \frac{(x+1)^2}{3\sqrt{x}} \right).$$

For part (b), taking natural logarithms yields

$$(x+1) \ln 5 = 2x \ln 3, \quad x = \frac{\ln 5}{2 \ln 3 - \ln 5}.$$

For part (c), the common domain is  $x > 2$ . Within this domain,

$$x(x-2) = 3x+4 \implies x^2 - 5x - 4 = 0 \implies x = \frac{5 \pm \sqrt{41}}{2}.$$

Only  $x = (5 + \sqrt{41})/2$  satisfies  $x > 2$ .

## 7. Radian measure and arc length.

- (a) Convert  $-315^\circ$  to radians and give a coterminal angle in  $[0, 2\pi)$ .
- (b) A wheel with radius 14 centimeters rotates through  $5\pi/7$  radians. Find the exact distance traveled by a point on its rim.

**Solution.** We have

$$-315^\circ \left( \frac{\pi}{180^\circ} \right) = -\frac{7\pi}{4}.$$

Adding  $2\pi$  gives the coterminal angle  $\pi/4$ . For the wheel,

$$s = r\theta = 14 \left( \frac{5\pi}{7} \right) = 10\pi \text{ centimeters.}$$

## 8. Unit-circle trigonometry.

Let  $\theta = 4\pi/3$ .

- (a) Identify the quadrant and reference angle of  $\theta$ .
- (b) Find the exact values of all six trigonometric functions at  $\theta$ .
- (c) Find an angle in  $[0, 2\pi)$  that is coterminal with  $-2\pi/3$ .

**Solution.** The angle lies in Quadrant III and has reference angle  $\pi/3$ . Its terminal point is

$$(\cos \theta, \sin \theta) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Therefore,

$$\sin \theta = -\frac{\sqrt{3}}{2}, \quad \cos \theta = -\frac{1}{2}, \quad \tan \theta = \sqrt{3},$$

and the reciprocal values are

$$\csc \theta = -\frac{2\sqrt{3}}{3}, \quad \sec \theta = -2, \quad \cot \theta = \frac{\sqrt{3}}{3}.$$

Adding  $2\pi$  to  $-2\pi/3$  gives the coterminal angle  $4\pi/3$ .

### 9. Trigonometric identities and equations.

- (a) If  $\cos \theta = -5/13$  and  $\theta$  lies in Quadrant II, find the exact values of  $\sin \theta$  and  $\tan \theta$ .
- (b) Solve  $2 \sin^2 x - \sin x = 0$  on  $[0, 2\pi)$ .

**Solution.** The Pythagorean identity gives

$$\sin^2 \theta = 1 - \frac{25}{169} = \frac{144}{169}.$$

Sine is positive in Quadrant II, so  $\sin \theta = 12/13$  and  $\tan \theta = -12/5$ .

For part (b), factor:

$$2 \sin^2 x - \sin x = \sin x(2 \sin x - 1) = 0.$$

Thus  $\sin x = 0$  or  $\sin x = 1/2$ . On  $[0, 2\pi)$ , the solutions are

$$x = 0, \quad \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \pi.$$

### 10. Graphs and inverse trigonometric functions.

- (a) For  $y = 3 \sin(2(x + \pi/4)) - 1$ , state the amplitude, period, horizontal shift, midline, and range.
- (b) Evaluate exactly:  $\arccos(-\sqrt{3}/2)$ ,  $\arctan(\tan(3\pi/4))$ , and  $\arcsin(\sin(5\pi/6))$ .
- (c) Simplify  $\sin(\arccos x)$  for  $-1 \leq x \leq 1$ .

**Solution.** The amplitude is 3, the period is  $2\pi/2 = \pi$ , the horizontal shift is left  $\pi/4$ , the midline is  $y = -1$ , and the range is  $[-4, 2]$ .

Using the principal-value ranges,

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}, \quad \arctan(\tan(3\pi/4)) = -\frac{\pi}{4}, \quad \arcsin(\sin(5\pi/6)) = \frac{\pi}{6}.$$

Finally, let  $\theta = \arccos x$ . Then  $\cos \theta = x$  and  $\theta \in [0, \pi]$ , where sine is nonnegative. Hence

$$\sin(\arccos x) = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2}.$$