

Homework 2

Math 140: Calculus with Analytic Geometry

Answer Key

1. Algebraic techniques for indeterminate limits.

Evaluate each limit. Show the algebraic step that removes the indeterminate form.

$$(a) \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3}{x}$$

$$(c) \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$$

Solution. Each expression initially has the indeterminate form 0/0.

For part (a), factor and cancel for $x \neq 3$:

$$\frac{x^2 - 5x + 6}{x^2 - 9} = \frac{(x-2)(x-3)}{(x-3)(x+3)} = \frac{x-2}{x+3}.$$

Therefore,

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x^2 - 9} = \frac{1}{6}.$$

For part (b), multiply by the conjugate:

$$\begin{aligned} \frac{\sqrt{9+x} - 3}{x} &= \frac{\sqrt{9+x} - 3}{x} \frac{\sqrt{9+x} + 3}{\sqrt{9+x} + 3} \\ &= \frac{x}{x(\sqrt{9+x} + 3)} = \frac{1}{\sqrt{9+x} + 3}, \quad x \neq 0. \end{aligned}$$

Thus the limit is 1/6.

For part (c), combine the fractions before taking the limit:

$$\begin{aligned} \frac{1}{x-2} - \frac{4}{x^2-4} &= \frac{x+2}{(x-2)(x+2)} - \frac{4}{(x-2)(x+2)} \\ &= \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}, \quad x \neq 2. \end{aligned}$$

Therefore, the limit is 1/4.

2. Limits of transcendental functions.

Evaluate each limit. Angles in trigonometric limits are measured in radians.

$$(a) \lim_{x \rightarrow 0} \frac{\sin(5x)}{3x}$$

$$(b) \lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)}{x - 1}$$

$$(c) \lim_{x \rightarrow 1} \ln\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution. For part (a), rewrite the expression to display the fundamental trigonometric limit:

$$\frac{\sin(5x)}{3x} = \frac{5}{3} \frac{\sin(5x)}{5x}.$$

Since $5x \rightarrow 0$, the limit is $5/3$.

For part (b), both the angle $x^2 - 1$ and the denominator approach 0. Factor $x^2 - 1$ and rewrite the expression to display the fundamental trigonometric limit:

$$\begin{aligned} \frac{\sin(x^2 - 1)}{x - 1} &= \frac{\sin(x^2 - 1)}{x^2 - 1} \cdot \frac{x^2 - 1}{x - 1} \\ &= \frac{\sin(x^2 - 1)}{x^2 - 1} (x + 1), \quad x \neq 1. \end{aligned}$$

As $x \rightarrow 1$, we have $x^2 - 1 \rightarrow 0$. Therefore, the first factor approaches 1 and the second approaches 2, so the limit is 2.

For part (c), the expression inside the logarithm is defined and positive for all x sufficiently close to 1 with $x \neq 1$. On those points,

$$\frac{x^2 - 1}{x - 1} = x + 1.$$

Because $\ln x$ is continuous for $x > 0$,

$$\lim_{x \rightarrow 1} \ln\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \rightarrow 1} \ln(x + 1) = \ln 2.$$

3. The Squeeze Theorem.

(a) Use the Squeeze Theorem to evaluate

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right).$$

Your solution must include explicit lower and upper bounds.

(b) For $0 < x < 1$, suppose the following inequalities are known:

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$$

Use these inequalities and the Squeeze Theorem to evaluate

$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3}.$$

Show clearly how you obtain the lower and upper bounds for the quotient.

Solution. For part (a), $|\sin(1/x)| \leq 1$, so

$$\left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|.$$

Equivalently,

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|.$$

Both outside functions approach 0 as $x \rightarrow 0$. The Squeeze Theorem gives

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0.$$

For part (b), subtract each part of the given inequality from x . Because subtraction reverses the order, we obtain

$$\frac{x^3}{6} - \frac{x^5}{120} \leq x - \sin x \leq \frac{x^3}{6}.$$

Since $x > 0$, division by x^3 preserves the inequalities:

$$\frac{1}{6} - \frac{x^2}{120} \leq \frac{x - \sin x}{x^3} \leq \frac{1}{6}.$$

Both outside expressions approach $1/6$ as $x \rightarrow 0^+$. Therefore, the Squeeze Theorem gives

$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3} = \frac{1}{6}.$$

4. Making a piecewise function continuous.

Find the values of a and b that make

$$f(x) = \begin{cases} x + a, & x < 0, \\ x^2 + b, & 0 \leq x < 2, \\ 3x - 1, & x \geq 2 \end{cases}$$

continuous for every real number. Justify your answer by checking continuity at both transition points.

Solution. Each formula is continuous on its own interval, so only $x = 0$ and $x = 2$ need to be checked.

At $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = a, \quad \lim_{x \rightarrow 0^+} f(x) = f(0) = b.$$

Continuity at 0 requires $a = b$.

At $x = 2$,

$$\lim_{x \rightarrow 2^-} f(x) = 4 + b, \quad \lim_{x \rightarrow 2^+} f(x) = f(2) = 5.$$

Continuity at 2 requires $4 + b = 5$, so $b = 1$. Since $a = b$, we also have $a = 1$.

5. The Intermediate Value Theorem.

Use the Intermediate Value Theorem to prove that the equation

$$\cos x = x$$

has at least one solution in the interval $(0, 1)$. Clearly identify the function to which you apply the theorem and verify every hypothesis. Does the Intermediate Value Theorem prove that the solution is unique? Explain.

Solution. Let

$$g(x) = \cos x - x.$$

Both $\cos x$ and x are continuous on $[0, 1]$, so their difference g is continuous on $[0, 1]$. Moreover,

$$g(0) = 1 > 0 \quad \text{and} \quad g(1) = \cos 1 - 1 < 0,$$

because $0 < 1 < \pi/2$ implies $\cos 1 < \cos 0 = 1$. Thus 0 lies between $g(0)$ and $g(1)$. By the Intermediate Value Theorem, there is some $c \in (0, 1)$ such that $g(c) = 0$. Therefore, $\cos c = c$.

The Intermediate Value Theorem proves only the existence of at least one solution; it does not prove uniqueness.