

Homework 3

Math 140: Calculus with Analytic Geometry

Answer Key

1. Secant lines and average rate of change.

Let $f(x) = x^2 - 2x$.

- (a) Find the average rate of change of f from $x = -1$ to $x = 2$.
- (b) Write an equation of the secant line through the corresponding points on the graph.

Solution. Since $f(-1) = 3$ and $f(2) = 0$,

$$m_{\text{sec}} = \frac{f(2) - f(-1)}{2 - (-1)} = -1.$$

Using $(2, 0)$, the secant line is $y = -(x - 2)$, or $y = -x + 2$.

2. The derivative at a point.

Let $f(x) = 1/(x + 1)$.

- (a) Simplify $\frac{f(1+h) - f(1)}{h}$ for $h \neq 0$.
- (b) Use the limit definition to find $f'(1)$.
- (c) Find an equation of the tangent line to $y = f(x)$ at $x = 1$.

Solution. For $h \neq 0$ and $h \neq -2$,

$$\frac{f(1+h) - f(1)}{h} = \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = -\frac{1}{2(2+h)}.$$

Thus $f'(1) = -1/4$. Since $f(1) = 1/2$, the tangent line is

$$y - \frac{1}{2} = -\frac{1}{4}(x - 1), \quad \text{or} \quad y = -\frac{1}{4}x + \frac{3}{4}.$$

3. The derivative as a function.

Starting from

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

find the derivative function for $f(x) = x^2 - 3x + 1$. Simplify the difference quotient before taking the limit.

Solution.

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h} \\&= \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3.\end{aligned}$$

4. Where derivatives do and do not exist.

For each function and specified input, determine whether the function is continuous and whether it is differentiable. Justify your conclusions using appropriate limits. Identify any vertical tangent.

(a) $f(x) = |x - 2|$ at $x = 2$.

(b) $g(x) = x^{2/3}$ at $x = 0$.

(c) $p(x) = \frac{x^2 - 1}{x - 1}$ at $x = 1$.

Solution. In part (a), f is continuous at 2, but

$$\frac{f(2+h) - f(2)}{h} = \frac{|h|}{h} = \begin{cases} -1, & h < 0, \\ 1, & h > 0. \end{cases}$$

The one-sided limits disagree, so $f'(2)$ does not exist; the graph has a corner.

In part (b), g is continuous at 0, but its difference quotient is

$$\frac{g(h) - g(0)}{h} = h^{-1/3}.$$

It approaches $-\infty$ from the left and $+\infty$ from the right. Thus $g'(0)$ does not exist as a finite number, and the graph has the vertical tangent $x = 0$.

In part (c), $p(1)$ is undefined, so p is neither continuous nor differentiable at 1. The graph has a hole at $(1, 2)$.

5. Linearity and the power rule.

Find the derivative of each function. Rewrite radicals and reciprocals as powers before differentiating.

(a) $f(x) = 4x^5 - 3x^2 + 7x - 9$.

(b) $g(x) = \sqrt{x} + \frac{3}{\sqrt{x}}$, where $x > 0$.

(c) $h(x) = 5x^{2/3} - 2x^{-3}$, where $x \neq 0$.

Solution.

$$\begin{aligned}f'(x) &= 20x^4 - 6x + 7, \\g'(x) &= \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-3/2} = \frac{x-3}{2x^{3/2}}, \quad x > 0, \\h'(x) &= \frac{10}{3}x^{-1/3} + 6x^{-4}, \quad x \neq 0.\end{aligned}$$

6. The product rule.

Differentiate each function. You do not need to expand your final answer unless expansion makes it simpler.

- (a) $f(x) = (x^2 + 1)(x^3 - 2x)$.
- (b) $g(x) = x^{2/3}(x^2 - 3)$, where $x \neq 0$.
- (c) $p(x) = x(x - 1)(x + 2)$.

Solution.

$$\begin{aligned}f'(x) &= 2x(x^3 - 2x) + (x^2 + 1)(3x^2 - 2) = 5x^4 - 3x^2 - 2, \\g'(x) &= \frac{2}{3}x^{-1/3}(x^2 - 3) + x^{2/3}(2x) = \frac{8x^2 - 6}{3x^{1/3}}, \quad x \neq 0, \\p'(x) &= (x - 1)(x + 2) + x(x + 2) + x(x - 1) = 3x^2 + 2x - 2.\end{aligned}$$

7. The quotient rule and simplifying first.

Find the derivative of each function. Preserve any domain restriction from the original formula.

- (a) $f(x) = \frac{x^3 - 2x}{x^2 + 1}$.
- (b) $g(x) = \frac{x^{5/2} + 2x^{3/2}}{\sqrt{x}}$, where $x > 0$. Simplify first.
- (c) $p(x) = \frac{(x - 1)(x^2 + 2)}{x - 1}$, where $x \neq 1$. Simplify first.

Solution.

$$f'(x) = \frac{(x^2 + 1)(3x^2 - 2) - (x^3 - 2x)(2x)}{(x^2 + 1)^2} = \frac{x^4 + 5x^2 - 2}{(x^2 + 1)^2}.$$

In part (b), $g(x) = x^2 + 2x$, so $g'(x) = 2x + 2$ for $x > 0$. In part (c), $p(x) = x^2 + 2$, so $p'(x) = 2x$ for $x \neq 1$.

8. Finding a tangent line.

Let $f(x) = (x^2 + 1)(x - 1)$.

- (a) Find $f'(x)$ using the product rule.
- (b) Find $f(1)$ and $f'(1)$.
- (c) Find an equation of the tangent line to the graph at $x = 1$.

Solution.

$$f'(x) = 2x(x - 1) + (x^2 + 1).$$

At $x = 1$, $f(1) = 0$ and $f'(1) = 2$. Therefore, the tangent line is

$$y = 2(x - 1) = 2x - 2.$$

9. Horizontal tangents and a specified slope.

Let $f(x) = x/(x^2 + 1)$.

- (a) Find $f'(x)$.
- (b) Find all points on the graph where the tangent line is horizontal.
- (c) Find all points on the graph where the tangent line has slope $-1/8$.

Solution.

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}.$$

For part (b), $f'(x) = 0$ at $x = \pm 1$, giving the points $(-1, -1/2)$ and $(1, 1/2)$.

For part (c),

$$\frac{1 - x^2}{(x^2 + 1)^2} = -\frac{1}{8} \iff (x^2 - 3)^2 = 0.$$

Thus $x = \pm\sqrt{3}$, giving

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right) \quad \text{and} \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right).$$

10. Reading a function from its derivative.

Let $F(x) = x^2(x - 3)$.

- (a) Find $F'(x)$ and write it in factored form.
- (b) Find all points on the graph of F where the tangent line is horizontal.
- (c) Determine where the graph of F is rising and where it is falling. Justify your answer from the sign of $F'(x)$.

Solution.

$$F'(x) = 2x(x - 3) + x^2 = 3x(x - 2).$$

The derivative is zero at $x = 0$ and $x = 2$. Since $F(0) = 0$ and $F(2) = -4$, the horizontal tangents occur at $(0, 0)$ and $(2, -4)$.

The sign of $3x(x - 2)$ is positive for $x < 0$, negative for $0 < x < 2$, and positive for $x > 2$. Thus, F is rising on $(-\infty, 0)$ and $(2, \infty)$ and falling on $(0, 2)$.