

Homework 4

Math 140: Calculus with Analytic Geometry

Answer Key

1. Combining derivative rules.

Differentiate each function. Retain any restrictions on its domain.

(a) $f(x) = \left(x^2 + \frac{1}{x}\right)(x^3 - 2x)$, where $x \neq 0$.

(b) $g(x) = \frac{x^2\sqrt{x+1}}{x-1}$, where $x > -1$ and $x \neq 1$.

Solution.

$$f'(x) = \left(2x - \frac{1}{x^2}\right)(x^3 - 2x) + \left(x^2 + \frac{1}{x}\right)(3x^2 - 2), \quad x \neq 0.$$

For g , the product, quotient, and chain rules give

$$g'(x) = \frac{2x\sqrt{x+1}}{x-1} + \frac{x^2}{2(x-1)\sqrt{x+1}} - \frac{x^2\sqrt{x+1}}{(x-1)^2}, \quad x > -1, \quad x \neq 1.$$

2. The chain rule in layers.

Find the derivatives.

(a) $f(x) = \sqrt{1 + (x^2 - 3x)^2}$.

(b) $g(x) = \sin^3(2x) - \cos(3x^2)$.

Solution. For (a), take $u = x^2 - 3x$, $v = 1 + u^2$, and $f = \sqrt{v}$. Then

$$f'(x) = \frac{(x^2 - 3x)(2x - 3)}{\sqrt{1 + (x^2 - 3x)^2}}.$$

In (b), $\sin^3(2x)$ means $(\sin(2x))^3$, and

$$g'(x) = 6\sin^2(2x)\cos(2x) + 6x\sin(3x^2).$$

3. Tangent lines and the chain rule.

Let $f(x) = \frac{x}{(x^2 + 1)^2}$.

(a) Find $f'(x)$ and write it as one fraction.

(b) Find equations of all horizontal tangent lines to the graph.

Solution. The product and chain rules give

$$f'(x) = (x^2 + 1)^{-2} - 4x^2(x^2 + 1)^{-3} = \frac{1 - 3x^2}{(x^2 + 1)^3}.$$

Thus $x = \pm 1/\sqrt{3}$; the corresponding y -values are $\pm 3\sqrt{3}/16$. The two horizontal lines are $y = 3\sqrt{3}/16$ and $y = -3\sqrt{3}/16$.

4. **Implicit differentiation on an ellipse.**

The curve $x^2 + xy + y^2 = 7$ contains $(1, 2)$.

- (a) Find dy/dx at every point where $x + 2y \neq 0$.
- (b) Find the tangent line at $(1, 2)$.
- (c) Determine all points on the curve where the tangent is horizontal.

Solution. Differentiation yields $2x + y + xy' + 2yy' = 0$, so

$$y' = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$, $y' = -4/5$, and the line is $y - 2 = -\frac{4}{5}(x - 1)$. A horizontal tangent requires $y = -2x$ and $x + 2y = -3x \neq 0$. Substitution gives $3x^2 = 7$, so the points are $(\sqrt{7/3}, -2\sqrt{7/3})$ and $(-\sqrt{7/3}, 2\sqrt{7/3})$.

5. **The folium of Descartes.**

Consider the curve $x^3 + y^3 = 6xy$ near $(3, 3)$.

- (a) Find an expression for dy/dx wherever the resulting denominator is nonzero.
- (b) Find the tangent line at $(3, 3)$.

Solution. From $3x^2 + 3y^2y' = 6y + 6xy'$,

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

At $(3, 3)$ the denominator is 3 and the slope is -1 ; the tangent line is $y - 3 = -(x - 3)$, or $x + y = 6$.

6. Exponential and logarithmic functions.

Find the derivative of each function on its domain.

(a) $f(x) = e^{x^2} \ln(1 + x^2)$.

(b) $g(x) = \ln\left(\frac{\sqrt{x+2}}{x^2+1}\right)$, where $x > -2$.

(c) $h(x) = x^x$, where $x > 0$. (Take logarithms first.)

Solution.

$$f'(x) = 2xe^{x^2} \left(\ln(1 + x^2) + \frac{1}{1 + x^2} \right).$$

Since $g(x) = \frac{1}{2} \ln(x+2) - \ln(x^2+1)$,

$$g'(x) = \frac{1}{2(x+2)} - \frac{2x}{x^2+1}, \quad x > -2.$$

Taking logarithms in $y = x^x$ gives $\ln y = x \ln x$, hence $y'/y = \ln x + 1$ and $h'(x) = x^x(\ln x + 1)$.

7. Deriving the derivative of the natural logarithm.

Use the identity $e^{\ln x} = x$ for $x > 0$.

(a) Differentiate both sides with respect to x and solve for $\frac{d}{dx} \ln x$. Explain why your division is valid.

(b) Use your result to find $\frac{d}{dx} \ln(x^2 + 4)$.

(c) Find the tangent line to $y = \ln(x^2 + 4)$ at $x = 1$.

Solution. By the chain rule, $e^{\ln x}(\ln x)' = 1$. Since $e^{\ln x} = x > 0$, division by x gives $(\ln x)' = 1/x$. Another application of the chain rule gives

$$\frac{d}{dx} \ln(x^2 + 4) = \frac{2x}{x^2 + 4}.$$

At $x = 1$, the slope is $2/5$ and the point is $(1, \ln 5)$. Thus the tangent line is $y - \ln 5 = \frac{2}{5}(x - 1)$.

8. Inverse trigonometric derivatives.

Differentiate the functions; state their derivative domains.

(a) $f(x) = \arcsin(2x - 1)$.

(b) $g(x) = \arctan(x^2) + \arccos(x/2)$.

Solution.

$$f'(x) = \frac{2}{\sqrt{1 - (2x - 1)^2}}, \quad 0 < x < 1.$$

Using the chain rule twice,

$$g'(x) = \frac{2x}{1 + x^4} - \frac{1}{2\sqrt{1 - x^2/4}}, \quad -2 < x < 2.$$

The functions themselves also have values at the relevant endpoints, but the displayed derivatives require the stated open intervals.

9. Higher derivatives and concavity.

Let $f(x) = xe^{-x}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find where $f''(x) = 0$ and determine the sign of f'' on either side.

(c) Find the equation of the tangent line at $x = 0$.

Solution. The product rule gives $f'(x) = e^{-x}(1 - x)$ and $f''(x) = e^{-x}(x - 2)$. Since $e^{-x} > 0$, f'' is negative for $x < 2$, zero at 2, and positive for $x > 2$. The graph is concave down for $x < 2$ and concave up for $x > 2$. As $f(0) = 0$ and $f'(0) = 1$, the tangent line is $y = x$.

10. Implicit second derivatives.

For the curve $x^2 + 4y^2 = 20$, work near the point $(2, 2)$, where y is a function of x .

(a) Find y' and y'' by differentiating the equation twice.

(b) Find $y''(2)$ and interpret its sign geometrically for this branch.

Solution. Differentiating once gives $2x + 8yy' = 0$, so $y' = -x/(4y)$. Differentiating again gives $2 + 8(y')^2 + 8yy'' = 0$, whence

$$y'' = -\frac{1 + 4(y')^2}{4y} = -\frac{x^2 + 4y^2}{16y^3} = -\frac{5}{4y^3}.$$

At $(2, 2)$, $y'' = -5/32$. The upper branch bends downward near $(2, 2)$: its tangent slopes decrease as x increases.