

Homework 5

Math 140: Calculus with Analytic Geometry

Name: _____

Due date: _____

Directions. Show enough work to support each answer. Give exact values unless an approximation is requested. For related rates, define your variables, draw a labeled sketch, and include units and the meaning of each rate's sign. Justify applications of theorems and support extrema with derivative signs or a comparison of function values. No calculators or other electronic devices are allowed.

1. Changing volumes.

A cylindrical tank has a fixed radius of 2 m and vertical sides. Water drains at $4\pi \text{ m}^3/\text{min}$ with no inflow. Let h be the water depth and V the volume of water. Assume the surface is horizontal and the tank is not empty.

- (a) Express V in terms of h . State dV/dt , including its sign.
- (b) Find dh/dt . Interpret its sign and include units.
- (c) Compare the water-level rates when $h = 1$ m and when $h = 2$ m. Explain your answer using the tank's shape.

2. Changing distances.

A boat is pulled toward a dock by a rope passing through a fixed ring 6 ft above the boat's attachment point. The rope between the ring and the boat is straight and shortens at 4 ft/s. The boat moves horizontally toward the dock, and its attachment point stays at a constant height. Let x be the horizontal distance from the attachment point to the point directly below the ring, and let s be the rope length between the ring and the boat.

- (a) Write an equation relating x and s . State ds/dt , including its sign.
- (b) Find dx/dt when $x = 8$ ft. How fast is the boat moving toward the dock?
- (c) At this instant, compare the boat's speed with the rope's shortening rate. Explain using your differentiated equation.

3. Changing water levels.

A trough has a constant triangular cross section and is 10 m long. Each end is an isosceles triangle with its vertex at the bottom, height 2 m, and top width 4 m. Water enters at $10 \text{ m}^3/\text{min}$ with no outflow. Let h be the water depth and w the width of the water surface. Assume $0 < h < 2$ and a horizontal water surface.

- (a) Use similar triangles to relate w and h . Express the water volume V in terms of h alone.
- (b) Find the rate at which the water level rises when $h = 1$ m.
- (c) With the same inflow, does the level rise faster at $h = 1/2$ m or at $h = 1$ m? Justify mathematically and explain in terms of the trough's shape.

4. Linear approximations and differentials.

Let $f(x) = x^{1/3}$ for $x > 0$.

- (a) Find the linearization $L(x)$ at $a = 8$ and use it to approximate $\sqrt[3]{8.12}$.
- (b) Set $dx = 0.12$ at $x = 8$. Find $dy = f'(8) dx$ and explain how it gives the same estimate as part (a). Write an exact expression for the actual change Δy .
- (c) Use the sign of f'' to decide whether the estimate in (a) is an overestimate or an underestimate, and whether dy is larger or smaller than Δy .

5. L'Hôpital's rule.

Evaluate each limit. State the indeterminate form before every application of L'Hôpital's rule and justify its use.

- (a) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{e^{2x} - 1}$.
- (b) $\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2}$.
- (c) $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$.

6. Rolle's Theorem.

- (a) For $f(x) = x^3 - 3x$ on $[-2, 1]$, verify every hypothesis of Rolle's Theorem and find all $c \in (-2, 1)$ with $f'(c) = 0$.
- (b) Let $g(x) = |x - 1|$ on $[0, 2]$. Although $g(0) = g(2)$, there is no $c \in (0, 2)$ with $g'(c) = 0$. Explain why this does not contradict Rolle's Theorem.

7. The Mean Value Theorem.

- (a) For $f(x) = \sqrt{x+1}$ on $[0, 8]$, verify the hypotheses of the Mean Value Theorem and find all $c \in (0, 8)$ satisfying its conclusion.
- (b) Suppose H is differentiable on $\mathbb{R}$ and $H'(x) > 0$ everywhere. Use the Mean Value Theorem to prove that H is strictly increasing: start with arbitrary $u < v$ and compare $H(v)$ with $H(u)$.

8. Derivative signs and horizontal tangents.

A differentiable function f on $\mathbb{R}$ satisfies

$$f'(x) = 15(x+1)x^2(x-1), \quad f(-1) = 2, \quad f(0) = 0, \quad f(1) = -2.$$

You do not need to find a formula for f .

- (a) Find all critical numbers and make a sign chart for f' . State intervals on which f is increasing or decreasing; you may list the intervals separated by critical numbers.
- (b) Classify each critical number as a local maximum, local minimum, or neither. Give each local extremum value and justify using derivative signs.
- (c) Find equations of all horizontal tangent lines. Which occurs without a local extremum?

9. Derivative signs and a domain break.

Let $g(x) = \frac{x^2}{x-2}$, where $x \neq 2$.

- (a) Find and factor $g'(x)$. Identify every critical number. Explain why $x = 2$ is not a critical number.
- (b) Make a sign chart for g' and state the intervals of increase and decrease. Identify every local maximum and local minimum, giving both location and value.
- (c) Find equations of all horizontal tangent lines. Explain why the two intervals of decrease separated by $x = 2$ cannot be combined into one interval of decrease.

10. Local and absolute extrema.

Let $q(x) = x^4 - 2x^2$ on $[-2, 1]$. Reserve the term local extremum for interior points of this interval.

- (a) Find all interior critical numbers. Factor $q'(x)$ and make a sign chart on the interval.
- (b) Identify every interior local maximum and local minimum, giving both its location and its value. Find equations of the horizontal tangent lines at the interior critical numbers.
- (c) Explain why the Extreme Value Theorem applies. Compare values at all interior critical numbers and both endpoints to find the absolute maximum and minimum. State every location where each is attained.