Implicit Differentiation

Math 140: Calculus with Analytic Geometry

Key Topics

- Implicitly defined functions
- Differentiating equations involving both x and y
- Applying the chain rule to expressions involving y
- Solving for $\frac{dy}{dx}$
- Tangent lines to implicitly defined curves
- Higher-order derivatives using implicit differentiation

1 Motivation

Thus far, we have differentiated functions written explicitly in the form $y = f(x)$. However, many curves arise from equations where y is not isolated on one side. For example,

$$x^2 + y^2 = 1, \qquad x^3 + xy + y^3 = 6.$$

To differentiate such equations, we use *implicit differentiation*.

2 Implicitly Defined Functions

Definition 2.1. *An equation involving both x and y is said to define y* implicitly *as a function of x if y is not explicitly solved in terms of x.*

Remark 2.1. *Even when y is not written as $y = f(x)$, it may still represent a function (or several functions) of x locally.*

3 The Idea of Implicit Differentiation

When differentiating implicitly, we treat y as a function of x and apply the chain rule whenever a derivative of y appears.

Remark 3.1. *Whenever you differentiate an expression involving y, multiply by $\frac{dy}{dx}$. For example,*

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}, \qquad \frac{d}{dx}(\sin(y)) = \cos(y)\frac{dy}{dx}.$$

4 Basic Examples

Example 4.1. *Differentiate the equation* $x^2 + y^2 = 1$ *with respect to* x*.*

Differentiate both sides:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1).$$

This gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Example 4.2. *Differentiate* $x^3 + xy + y^3 = 6$*.*

Differentiate both sides:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6).$$

Compute each term:

$$3x^2 + \left(x\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} = 0.$$

Group terms involving $\frac{dy}{dx}$:

$$(x + 3y^2)\frac{dy}{dx} = -(3x^2 + y).$$

Thus,

$$\frac{dy}{dx} = -\frac{3x^2 + y}{x + 3y^2}.$$

5 Trigonometric Example

Example 5.1. *Differentiate* $\sin(xy) = x$*.*

Differentiate both sides:

$$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(x).$$

Apply the chain rule to the left side:

$$\cos(xy)\frac{d}{dx}(xy) = 1.$$

Since $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$, we obtain

$$\cos(xy)\left(x\frac{dy}{dx} + y\right) = 1.$$

Solve for $\frac{dy}{dx}$:

$$x\cos(xy)\frac{dy}{dx} = 1 - y\cos(xy), \qquad \frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}.$$

6 Tangent Line via Implicit Differentiation

Example 6.1. *Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.*

From implicit differentiation,

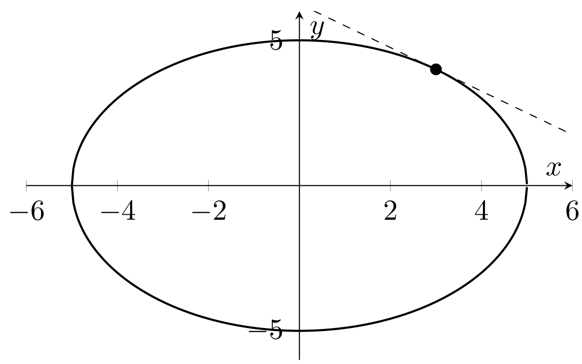
$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

At $(3, 4)$,

$$\frac{dy}{dx} = -\frac{3}{4}.$$

The tangent line is

$$y - 4 = -\frac{3}{4}(x - 3).$$



7 Second Derivatives via Implicit Differentiation

Implicit differentiation can be applied repeatedly to compute higher-order derivatives.

Example 7.1. *For the curve $x^2 + y^2 = 1$, find $\frac{d^2y}{dx^2}$.*

From the first derivative,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Differentiate again using the quotient rule:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}.$$

Substitute $\frac{dy}{dx} = -\frac{x}{y}$:

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{x^2 + y^2}{y^3}.$$

Since $x^2 + y^2 = 1$ on the curve,

$$\frac{d^2y}{dx^2} = -\frac{1}{y^3}.$$

8 Common Pitfalls

Remark 8.1.

- *Forgetting to multiply by* $\frac{dy}{dx}$ *when differentiating expressions involving* y.
- *Forgetting the product rule when differentiating terms such as* xy.
- *For tangent line problems, forgetting to evaluate* $\frac{dy}{dx}$ *at the given point.*

9 Why This Matters for Calculus

Implicit differentiation allows us to analyze curves defined by equations rather than explicit formulas.

- Many important curves, such as circles and ellipses, are naturally defined implicitly.
- It extends the chain rule to more general situations.
- It is essential for related rates, optimization with constraints, and curve sketching.
- It prepares us for inverse functions and logarithmic differentiation.