Derivatives of Inverse Trigonometric Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- Inverse trigonometric functions as mappings from ratios to angles
- Implicit differentiation of inverse trigonometric functions
- Right–triangle interpretations of inverse trigonometric derivatives
- Derivatives of $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$
- Combining inverse trigonometric derivatives with previously learned rules
- Tangent lines to curves involving inverse trigonometric functions

1 Motivation

Inverse trigonometric functions arise when solving trigonometric equations and appear frequently later in the course when evaluating integrals. Their derivatives combine implicit differentiation with geometric reasoning using right triangles.

2 Inverse Trigonometric Functions and Their Derivatives

Definition 2.1. *Inverse trigonometric functions map* ratios *to* angles*.*

- $y = \arcsin(x)$ *means* $\sin(y) = x$*, where* $-1 \leq x \leq 1$ *and* $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $y = \arccos(x)$ *means* $\cos(y) = x$*, where* $-1 \leq x \leq 1$ *and* $0 \leq y \leq \pi$
- $y = \arctan(x)$ *means* $\tan(y) = x$*, where* $-\infty < x < \infty$ *and* $-\frac{\pi}{2} < y < \frac{\pi}{2}$

In each case, the inverse trigonometric function takes a ratio of sides of a right triangle and returns the corresponding angle.

Theorem 2.1.

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

Proof. Let $y = \arcsin(x)$, so $\sin(y) = x$. Differentiating,

$$\cos(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}.$$

Since $\sin(y) = \dfrac{x}{1}$, choose a right triangle with opposite side x and hypotenuse 1.

Thus $\cos(y) = \sqrt{1-x^2}$, and the result follows. □

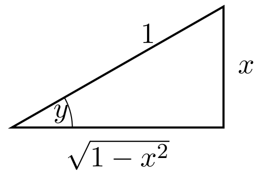

Figure 1: Right triangle interpretation of $\sin(y) = x$.

Theorem 2.2.

$$\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

Proof. Let $y = \arccos(x)$ so that $\cos(y) = x$. Differentiating,

$$-\sin(y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin(y)}.$$

From the same triangle, $\sin(y) = \sqrt{1-x^2}$. □

Theorem 2.3.

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}, \qquad -\infty < x < \infty.$$

Proof. Let $y = \arctan(x)$ so $\tan(y) = x$. Differentiating,

$$\sec^2(y)\frac{dy}{dx} = 1.$$

Choose a right triangle with adjacent side 1 and opposite side x.

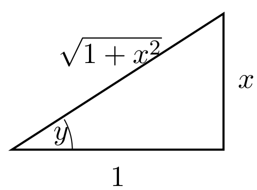

Figure 2: Right triangle interpretation of $\tan(y) = x$.

Then $\sec^2(y) = 1 + x^2$, giving the result. □

3 Examples

Example 3.1. *Differentiate* $f(x) = \arcsin(3x)$.

$$f'(x) = \frac{3}{\sqrt{1-9x^2}}.$$

Example 3.2. *Differentiate* $g(x) = x\arctan(x)$.

$$g'(x) = \arctan(x) + \frac{x}{1+x^2}.$$

Example 3.3. *Differentiate* $h(x) = \dfrac{\arccos(x)}{x}$.

$$h'(x) = \frac{-x/\sqrt{1-x^2} - \arccos(x)}{x^2}.$$

Example 3.4. *Differentiate* $p(x) = \big(\arctan(x)\big)^2$.

$$p'(x) = \frac{2\arctan(x)}{1+x^2}.$$

Example 3.5. *Find the equation of the tangent line to* $y = \arcsin(x)$ *at* $x = 0$.

$$y' = \frac{1}{\sqrt{1-x^2}}, \qquad y'(0) = 1.$$

The point is $(0, 0)$, so the tangent line is $y = x$.

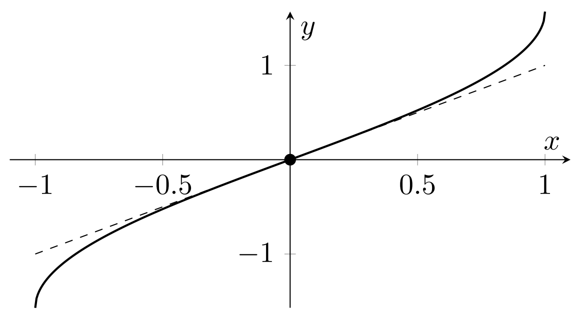

Figure 3: The graph of $y = \arcsin(x)$ together with its tangent line at $x = 0$.

4 Why This Matters for Calculus

Inverse trigonometric derivatives complete the differentiation toolkit.

- They provide explicit derivatives for inverse functions.
- Their geometric derivations reinforce right–triangle intuition.
- They appear frequently in integration and substitution techniques.
- They connect inverse functions, implicit differentiation, and the chain rule.
- They are essential for solving trigonometric equations involving inverses.