L'Hôpital's Rule

Math 140: Calculus with Analytic Geometry

Key Topics

- Indeterminate forms
- L'Hôpital's Rule for $\frac{0}{0}$ and $\frac{\infty}{\infty}$
- Repeated application of L'Hôpital's Rule
- Converting other indeterminate forms
- Indeterminate powers

1 Motivation

Consider the limit

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Direct substitution gives

$$\frac{0}{0},$$

which is called an *indeterminate form*. It does not tell us the value of the limit.

Other common indeterminate forms include:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0.$$

Today we develop a powerful method for resolving many of these limits.

2 L'Hôpital's Rule

Theorem 2.1 ($\frac{0}{0}$ Case)

Let f and g be differentiable on an open interval containing a, except possibly at a. Suppose

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0,$$

and suppose $g'(x) \neq 0$ near a.
If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists (or is $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Theorem 2.2 ($\frac{\infty}{\infty}$ Case)

If

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty,$$

and the other hypotheses above hold, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Important Remarks

- L'Hôpital's Rule applies only to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms.
- You must verify the indeterminate form before applying the rule.
- The rule may be applied more than once if necessary.

3 Examples

Example 3.1

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Substitution gives $\frac{0}{0}$.
Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Example 3.2

$$\lim_{x \to 0} \frac{1 - \cos x}{x}.$$

Substitution gives $\frac{0}{0}$.
Differentiate:

$$\lim_{x \to 0} \frac{\sin x}{1} = 0.$$

Example 3.3 (Repeated Application)

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}.$$

Substitution gives $\frac{0}{0}$.
Differentiate:

$$\frac{1 - \cos x}{3x^2}.$$

Still $\frac{0}{0}$.
Differentiate again:

$$\frac{\sin x}{6x}.$$

Still $\frac{0}{0}$.
Differentiate again:

$$\frac{\cos x}{6}.$$

Thus,

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1}{6}.$$

Example 3.4 ($\frac{\infty}{\infty}$)

$$\lim_{x\to\infty}\frac{\ln x}{x}.$$

Substitution gives $\frac{\infty}{\infty}$.
Differentiate:

$$\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

Thus, $\ln x$ grows more slowly than x.

4 Other Indeterminate Forms

$0\cdot\infty$

Rewrite as a quotient.
Example:

$$\lim_{x\to 0^+} x\ln x.$$

Rewrite:

$$x\ln x=\frac{\ln x}{1/x}.$$

Now we have $\frac{-\infty}{\infty}$.
Apply L'Hôpital:

$$\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0.$$

$\infty-\infty$

Combine expressions into a single fraction before applying L'Hôpital's Rule.

Indeterminate Powers

Expressions of the form

$$0^0, \quad 1^\infty, \quad \infty^0$$

are indeterminate powers.

Example 4.1

$$\lim_{x\to 0^+} x^x.$$

As $x \to 0^+$,

$$x^x \to 0^0,$$

which is indeterminate.
Let

$$y = x^x.$$

Take natural logarithms:

$$\ln y = x \ln x.$$

Now evaluate

$$\lim_{x\to 0^+} x \ln x.$$

Rewrite:

$$x \ln x = \frac{\ln x}{1/x}.$$

This is $\frac{-\infty}{\infty}$.
Apply L'Hôpital:

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Thus,

$$\lim_{x\to 0^+} \ln y = 0.$$

Exponentiate:

$$\lim_{x \to 0^+} y = e^0 = 1.$$

Therefore,

$$\boxed{\lim_{x \to 0^+} x^x = 1.}$$

5 Geometric Insight

Suppose

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

produces the indeterminate form $\frac{0}{0}$. That is,

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0.$$

Notice that this statement concerns the *limits* of f and g as x approaches a. It does not require that $f(a) = 0$ or $g(a) = 0$, nor does it require that f or g even be defined at a.

Because f and g are differentiable near a, we can approximate them for x close to a using their linearizations:

$$f(x) \approx f(a) + f'(a)(x - a),$$
$$g(x) \approx g(a) + g'(a)(x - a).$$

Now, since $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, the dominant behavior of both functions near a must vanish.

If f and g are continuous at a, then $f(a) = 0$ and $g(a) = 0$, and the linear approximations simplify to

$$f(x) \approx f'(a)(x - a), \qquad g(x) \approx g'(a)(x - a).$$

Even if f and g are not defined at a, their behavior near a is governed by these linear terms.

Thus, for x sufficiently close to a (with $x \neq a$),

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)}.$$

Canceling $(x - a)$ gives

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)}.$$

This explains why the limit of the ratio of the functions is determined by the ratio of their derivatives.

The same reasoning applies in the $\frac{\infty}{\infty}$ case: when both functions grow without bound, the dominant growth rate is governed by their derivatives.

6 Why This Matters

L'Hôpital's Rule:

- Resolves many difficult limits efficiently.
- Reveals relative growth rates of functions.
- Connects derivatives and limits in a natural way.
- Extends the idea of linear approximation to limits.