The Limit Definition of the Derivative

Math 140: Calculus with Analytic Geometry

Key Topics

- The derivative as a limit of difference quotients
- Differentiability at a point
- Computing derivatives from first principles
- Differentiability versus continuity

1 From Secant Slopes to a Limit

Recall that the slope of the secant line to $y = f(x)$ through $(a, f(a))$ and $(a+h, f(a+h))$ is

$$\frac{f(a+h) - f(a)}{h}.$$

As $h \to 0$, the secant line approaches the tangent line at $x = a$. This motivates the following definition.

2 The Derivative at a Point

Definition 2.1. *Let f be a function defined on an open interval containing a. The* ***derivative of f at a****, denoted $f'(a)$, is*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

Definition 2.2. *If the derivative $f'(a)$ exists, we say that f is* ***differentiable*** *at $x = a$.*

3 Computing Derivatives from First Principles

3.1 Linear Example

Example 3.1. *Let $f(x) = 3x - 5$. Compute $f'(a)$.*

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{3(a+h) - 5 - (3a - 5)}{h} \\ &= \lim_{h \to 0} \frac{3h}{h} = 3. \end{aligned}$$

3.2 Quadratic Example

Example 3.2. *Let $f(x) = x^2 + 3x + 2$. Compute $f'(a)$ using the limit definition.*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^2 + 3(a+h) + 2 - (a^2 + 3a + 2)}{h} \\ &= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 + 3a + 3h + 2 - a^2 - 3a - 2}{h} \\ &= \lim_{h\to 0} \frac{2ah + h^2 + 3h}{h} \\ &= \lim_{h\to 0} (2a + h + 3) \\ &= 2a + 3. \end{aligned}$$

3.3 Square Root Example

Example 3.3. *Let $f(x) = \sqrt{x}$. Compute $f'(a)$ for $a > 0$.*

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \\ &= \lim_{h\to 0} \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}. \end{aligned}$$

4 Differentiability and Continuity

Theorem 4.1 (Theorem 5.1)**.** *If a function f is differentiable at $x = a$, then f is continuous at $x = a$.*

Proof. Assume f is differentiable at a. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. Rewrite

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) h.$$

Taking limits as $h \to 0$ gives

$$\lim_{h\to 0} [f(a+h) - f(a)] = \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \right) \left(\lim_{h\to 0} h \right) = f'(a) \cdot 0 = 0.$$

Thus,

$$\lim_{x\to a} f(x) = f(a),$$

so f is continuous at $x = a$. □

4.1 Continuity Without Differentiability

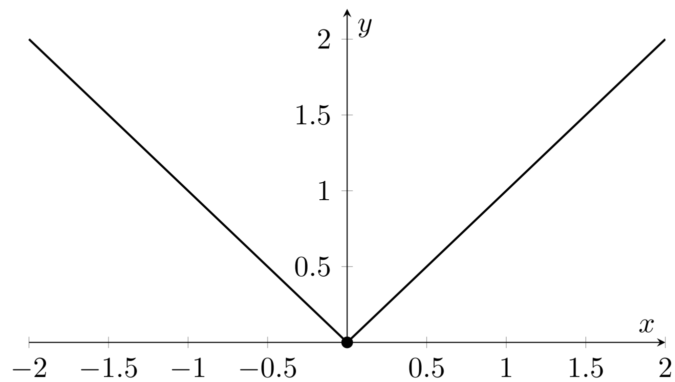
Example 4.1. *Define $f(x) = |x|$. The function is continuous at $x = 0$ since*

$$\lim_{x \to 0} |x| = 0 = f(0).$$

However, the derivative at 0 *does not exist because*

$$\lim_{h \to 0^+} \frac{|h| - 0}{h} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{|h| - 0}{h} = -1.$$

Since the one-sided limits are not equal, f is not differentiable at $x = 0$.



5 Why This Matters for Calculus

- The limit definition provides a rigorous foundation for derivative rules.
- Differentiability is a stronger condition than continuity.
- Understanding where derivatives fail helps explain corners and cusps in graphs.