Limits at Infinity

Math 140: Calculus with Analytic Geometry

1 Motivation

In previous sections we studied limits as x approaches a finite number. We now investigate the behavior of functions as x becomes very large or very negative.

Questions of this type include:

- What happens to a function as $x \to \infty$?
- Does the graph approach a horizontal line?
- How do different functions grow compared to one another?

These questions lead to the idea of **limits at infinity** and **horizontal asymptotes**.

2 Limits at Infinity

Definition 2.1

Let f be a function defined for large values of x. We write

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ become arbitrarily close to L as x becomes large. Similarly,

$$\lim_{x\to-\infty} f(x) = L$$

if $f(x)$ approaches L as x becomes very negative.

3 Horizontal Asymptotes

Definition 3.1
The line

$$y = L$$

is called a **horizontal asymptote** of the graph of f if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

This means the graph approaches the horizontal line $y = L$ as x becomes very large (or very negative).

4 Example: A Rational Function

Evaluate

$$\lim_{x \to \infty} \frac{3x+1}{x+2}.$$

Solution
Divide numerator and denominator by x:

$$\frac{3x+1}{x+2} = \frac{3+\frac{1}{x}}{1+\frac{2}{x}}.$$

Now take the limit:

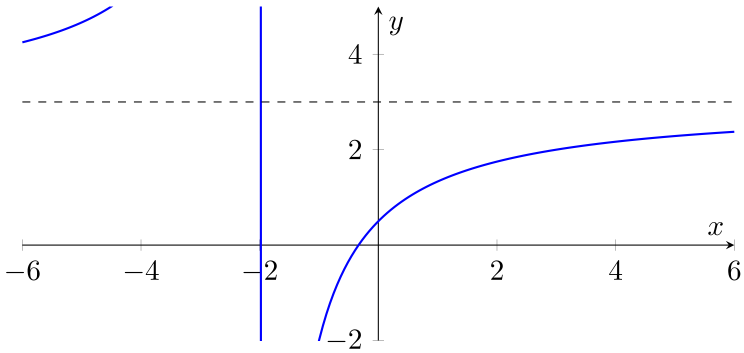
$$\lim_{x \to \infty} \frac{3+\frac{1}{x}}{1+\frac{2}{x}} = \frac{3+0}{1+0} = 3.$$

Thus

$$\lim_{x \to \infty} \frac{3x+1}{x+2} = 3.$$

Therefore the graph has the horizontal asymptote

$$y = 3.$$



5 Example Using L'Hôpital's Rule

Evaluate

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Solution
Direct substitution gives

$$\frac{\infty}{\infty},$$

which is an indeterminate form. Therefore we apply L'Hôpital's Rule.
Differentiate numerator and denominator:

$$\lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x}.$$

Thus

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

This result shows that x grows faster than $\ln x$.

6 Example: Exponential Growth

Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^3}.$$

Solution

Direct substitution gives

$$\frac{\infty}{\infty}.$$

Apply L'Hôpital's Rule repeatedly:

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6}.$$

Thus

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \infty.$$

This shows that exponential functions grow faster than polynomial functions.

7 Growth Rates of Functions

The previous examples illustrate an important hierarchy of growth rates:

$$\ln x \ll x^p \ll e^x$$

for any power $p > 0$.

In words:

- Logarithmic functions grow slowest.
- Polynomial functions grow faster.
- Exponential functions grow fastest.

8 Why This Matters

Limits at infinity help us understand the long-term behavior of functions.

They allow us to:

- determine horizontal asymptotes,
- compare growth rates of functions,
- analyze rational and exponential models,
- understand the global shape of graphs.

These ideas will play an important role when we study curve sketching and optimization problems.