Finding Limits Graphically

Math 140: Calculus with Analytic Geometry

Key Topics

- The intuitive meaning of a limit
- Estimating limits from graphs
- One-sided limits
- The ε–δ definition of a limit

1 The Idea of a Limit

The limit of a function describes the behavior of the function values as the input approaches a particular number. The key idea is that limits describe what happens *near* a point, not necessarily *at* the point.

We denote the limit of a function f as x approaches c by

$$\lim_{x \to c} f(x).$$

If this limit exists, it represents the value that $f(x)$ approaches as x gets arbitrarily close to c.

Remark. *The value of $f(c)$ plays no role in determining $\lim_{x \to c} f(x)$. Limits depend only on the behavior of $f(x)$ for values of x close to c.*

2 Limits from Graphs

When evaluating limits graphically, we ask:

> As x approaches c from the left and from the right, what y-value does the graph approach?

2.1 Example: A limit exists even though the function value differs

Consider the piecewise function

$$f(x) = \begin{cases} x^2, & x < 0, \\ 1, & x = 0, \\ x, & x > 0. \end{cases}$$

We determine $\lim_{x \to 0} f(x)$ from the graph shown in Figure 1.

From the left, $f(x) = x^2 \to 0$ as $x \to 0^-$. From the right, $f(x) = x \to 0$ as $x \to 0^+$.

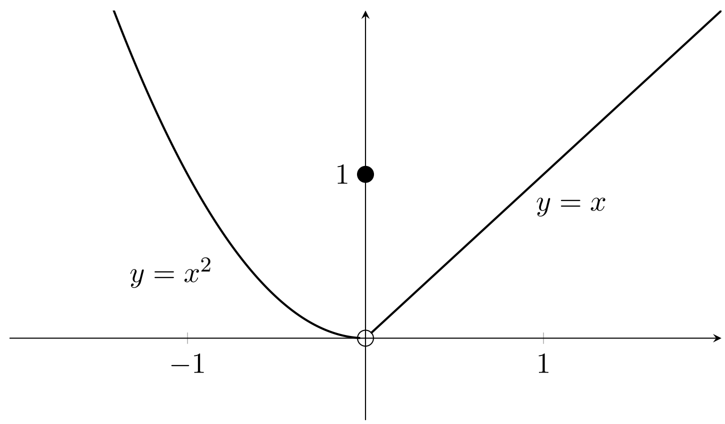

Figure 1: A function whose values approach 0 as $x \to 0$ even though $f(0) = 1$.

Since both one-sided limits agree,

$$\lim_{x \to 0} f(x) = 0,$$

even though $f(0) = 1$.

In some cases, the behavior of a function depends on the direction from which x approaches a point.

$$\lim_{x \to c^-} f(x) \quad \text{(left-hand limit)}, \qquad \lim_{x \to c^+} f(x) \quad \text{(right-hand limit)}.$$

Remark. *The limit* $\lim_{x \to c} f(x)$ *exists if and only if both one-sided limits exist and are equal.*

2.2 Example: One-sided limits disagree

Figure 2 shows a function whose left-hand and right-hand limits at $x = 1$ are different.

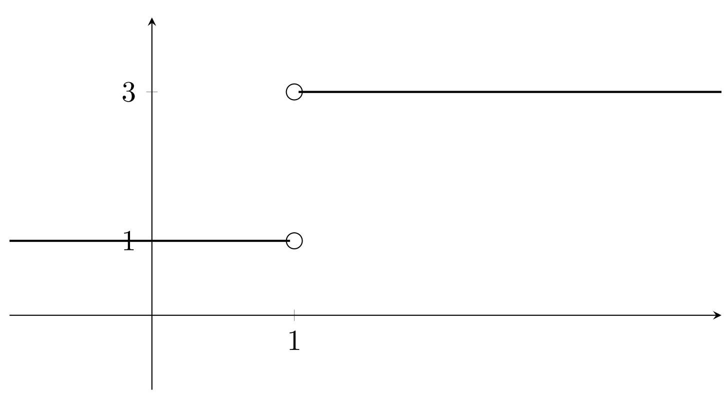

Figure 2: A function for which the left-hand and right-hand limits at $x = 1$ are unequal.

From the graph,

$$\lim_{x \to 1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 3.$$

Since these limits are not equal, $\lim_{x \to 1} f(x)$ does not exist.

3 The ε–δ Definition of a Limit

Definition. *We say that*

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

3.1 Geometric interpretation

Figure 3 illustrates the geometric meaning of the ε–δ definition. Note that if $x \neq c$ lies in the interval $(c - \delta, c + \delta)$, then $f(x)$ must lie in the interval $(L - \epsilon, L + \epsilon)$; in layman terms, if x is no further than δ away from c (and not equal to c), then $f(x)$ is no further than ϵ away from L.

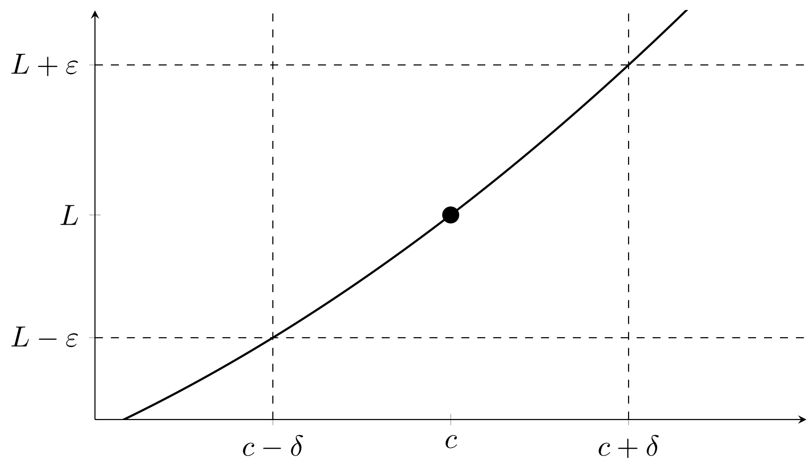

Figure 3: A geometric illustration of the ε–δ definition of a limit.

3.2 Example using the definition

We show that

$$\lim_{x \to 2} (3x + 1) = 7.$$

Proof. Let $\varepsilon > 0$ be given. We seek $\delta > 0$ such that

$$0 < |x - 2| < \delta \quad \implies \quad |(3x + 1) - 7| < \varepsilon.$$

Since

$$|(3x + 1) - 7| = 3|x - 2|,$$

it suffices to require $|x - 2| < \varepsilon/3$. Choose

$$\delta = \frac{\varepsilon}{3}.$$

Then whenever $0 < |x - 2| < \delta$, we have $|(3x + 1) - 7| < \varepsilon$, completing the proof. □