Linear Approximations and Differentials

Math 140: Calculus with Analytic Geometry

Key Topics

- Linearization (tangent line approximation)
- Local behavior of a function near a point
- Differentials dx and dy and their interpretation
- Estimating change and propagating measurement error
- Concavity and the direction of tangent line error

1 Motivation

In applications, we often need quick estimates without a calculator, or we need to understand how a small change in an input affects the output.

Examples:

- Approximate $\sqrt{100.8}$ without computing the square root exactly.
- Estimate how much the volume of a sphere changes when the radius measurement is off by a small amount.
- Approximate $\ln(1.01)$ using simple arithmetic.

The main idea is that a differentiable function looks almost linear when you zoom in near a point.

2 Linear Approximation (Linearization)

Let f be differentiable at $x = a$. The tangent line to $y = f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

We use $L(x)$ to approximate $f(x)$ when x is close to a.

Definition 2.1 (Linearization)

The *linearization* of f at $x = a$ is the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For x near a, we often write

$$f(x) \approx L(x).$$

Remark 2.1. This is a local approximation: it is intended for values of x close to a. The closer x is to a, the better the approximation typically becomes.

A picture: function vs. tangent line (exact tangency)

In the figure below we use the concave up function

$$f(x) = \frac{x^2}{4} + 1,$$

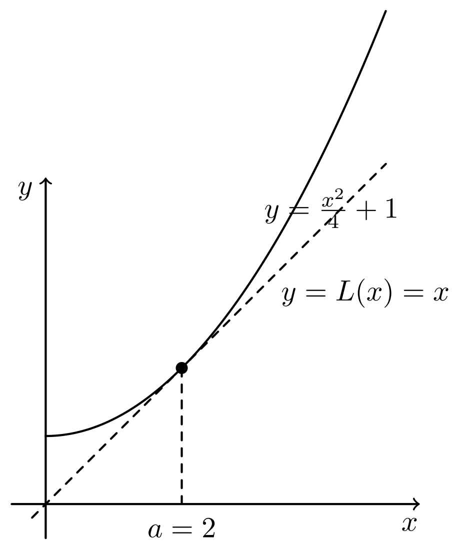
and we take $a = 2$. Then $f(2) = 2$ and $f'(x) = \frac{x}{2}$, so $f'(2) = 1$. Hence the tangent line at $x = 2$ is

$$L(x) = 2 + 1(x - 2) = x.$$

This line intersects the curve at exactly one point (the point of tangency).

3 Differentials

Linearization is often expressed using differentials, which are convenient for estimating small changes.



Suppose $y = f(x)$. If x changes from a to $a + \Delta x$, the actual change in y is

$$\Delta y = f(a + \Delta x) - f(a).$$

The differential dy is the linear estimate of Δy:

$$dy = f'(a)\, dx,$$

where we interpret dx as a small change in x (often $dx = \Delta x$ in computations).

Definition 3.1 (Differential)

Let $y = f(x)$ and suppose f is differentiable at $x = a$. The *differential* of y at $x = a$ is

$$dy = f'(a)\, dx.$$

Remark 3.1. For small changes, we use

$$\Delta y \approx dy = f'(a)\, \Delta x.$$

This is exactly the same approximation as $f(a + \Delta x) \approx f(a) + f'(a)\Delta x$.

4 Examples

Example 4.1: Approximating a square root

Approximate $\sqrt{4.8}$.

Let $f(x) = \sqrt{x}$ and choose $a = 4$ (a nearby perfect square). Then $f(4) = 2$ and

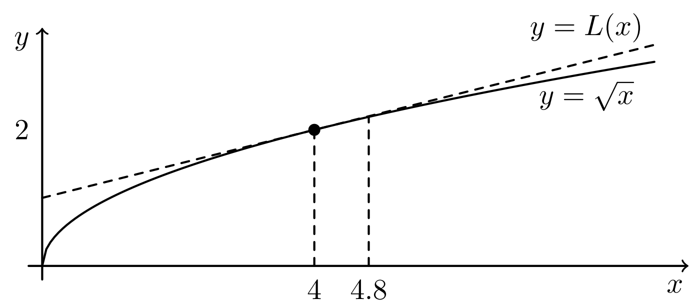
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(4) = \frac{1}{4}.$$

The linearization at $x = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

With $\Delta x = 0.8$,

$$\sqrt{4.8} \approx 2 + \frac{1}{4}(0.8) = 2 + 0.2 = 2.2.$$

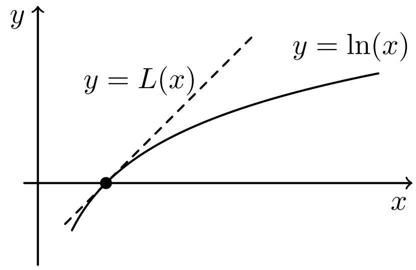


Example 4.2: Approximating $\ln(1.01)$

Approximate $\ln(1.01)$.

Let $f(x) = \ln(x)$ and choose $a = 1$. Then $f(1) = 0$ and $f'(x) = \frac{1}{x}$, so $f'(1) = 1$. With $\Delta x = 0.01$,

$$\ln(1.01) = f(1.01) \approx f(1) + f'(1)(0.01) = 0 + 1(0.01) = 0.01.$$



Example 4.3: Using differentials to estimate the error in the volume of a sphere

A sphere has radius $r = 10$ cm. Suppose the radius measurement may be off by ± 0.1 cm. Estimate the resulting error in the volume.

The volume is

$$V = \frac{4}{3}\pi r^3.$$

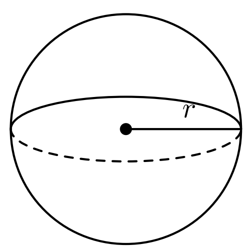
Differentiate:

$$dV = 4\pi r^2\,dr.$$

At $r = 10$ and $dr = \pm 0.1$,

$$dV \approx 4\pi(10)^2(0.1) = 40\pi \text{ cm}^3.$$

So the volume error is approximately $\pm 40\pi$ cm^3.



$dV \approx 4\pi r^2\,dr$

Example 4.4: Relative error estimate

Continue the previous example. Estimate the *relative* error in volume.

The relative error is approximately

$$\frac{dV}{V}.$$

Using $V = \frac{4}{3}\pi r^3$ and $dV = 4\pi r^2\,dr$,

$$\frac{dV}{V} \approx \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} = \frac{3\,dr}{r}.$$

With $r = 10$ and $dr = 0.1$,

$$\frac{dV}{V} \approx \frac{3(0.1)}{10} = 0.03.$$

So the volume has about a 3% relative error.

5 Accuracy and Error

Linearization is exact for linear functions, and it is accurate for small Δx because the tangent line matches both the value and the slope of the function at $x = a$.

Concavity tells us the *direction* of the error. Let f be differentiable on an interval I.

- If f is *concave up* on I, then for every $a \in I$ and every $x \in I$,

  $$f(x) \geq f(a) + f'(a)(x - a).$$

  That is, the graph of f lies *above* every tangent line on I.

- If f is *concave down* on I, then for every $a \in I$ and every $x \in I$,

  $$f(x) \leq f(a) + f'(a)(x - a).$$

  That is, the graph of f lies *below* every tangent line on I.

Remark 5.1. In particular, if f is concave up near a and x is close to a, then the tangent line approximation $L(x)$ typically *underestimates* $f(x)$. If f is concave down near a, then $L(x)$ typically *overestimates* $f(x)$.

6 Why This Matters for Calculus

Linear approximations and differentials are a bridge between pure calculus and applications.

- They provide fast estimates for complicated functions.
- They convert small input changes into approximate output changes.
- They formalize measurement error propagation in formulas.
- They prepare us for Newton's method and more advanced approximation techniques.