

Linear Approximations, Differentials, and L'Hôpital's Rule

Math 140: Calculus with Analytic Geometry

1 Linear Approximation: Using a Tangent Line

1.1 Predicting a small change

Example 1.1 (A growing square). A square's side increases from 3 cm to 3.1 cm. How much does its area increase?

For $A(x) = x^2$, the actual and predicted increases are

$$\Delta A = (3.1)^2 - 3^2 = 0.61 \text{ cm}^2, \quad A'(3)(0.1) = 6(0.1) = 0.6 \text{ cm}^2.$$

The derivative $A'(3) = 6$ has units cm^2/cm ; multiplying by the side-length change gives an area change. For a side length a and an increase $h > 0$, the added area is

$$(a + h)^2 - a^2 = 2ah + h^2.$$

The two strips have total area $2ah$. The corner has area h^2 , which is the error in the derivative's prediction.

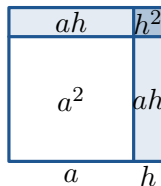


Figure 1: The strips give the linear change; the corner gives the error.

1.2 The linearization formula

For a differentiable function, the derivative definition gives

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$

For small h , this suggests $f(a + h) - f(a) \approx f'(a)h$. Hence

$$f(a + h) \approx f(a) + f'(a)h.$$

Writing $x = a + h$, we obtain the *linearization* at a :

$$L(x) = f(a) + f'(a)(x - a).$$

This is the tangent line. It matches both the value and slope of f at a , so it gives a simple local model of the function.

1.3 Choosing a useful starting point

Choose a near the desired input where $f(a)$ and $f'(a)$ are easy to calculate. The approximation is local: it need not work well far from a .

Example 1.2 (Estimating a square root). Approximate $\sqrt{4.08}$ using $f(x) = \sqrt{x}$ at $a = 4$.

Since $f(4) = 2$ and $f'(4) = 1/4$,

$$L(x) = 2 + \frac{1}{4}(x - 4), \quad \sqrt{4.08} \approx L(4.08) = 2.02.$$

The input increase 0.08 predicts an output increase 0.02.

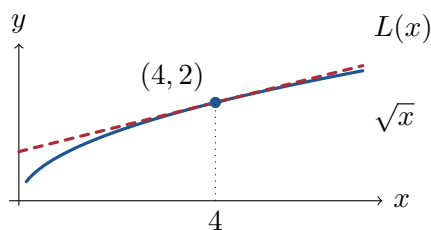


Figure 2: The tangent line lies above $\sqrt{x}$.

1.4 Concavity and accuracy

On an interval of concavity, tangent lines underestimate a concave up function and overestimate a concave down function. For $f(x) = \sqrt{x}$, $f''(x) = -1/(4x^{3/2}) < 0$ for $x > 0$, so 2.02 is an overestimate. Concavity tells us the direction of the error, not its size.

Example 1.3 (Estimating a logarithm). For $f(x) = \ln x$ at $a = 1$, $f(1) = 0$ and $f'(1) = 1$. Therefore

$$L(x) = x - 1, \quad \ln(1.03) \approx 0.03.$$

This is an overestimate because the logarithm is concave down.

Example 1.4 (A useful small-angle approximation). For $f(x) = \sin x$ at $a = 0$, $f(0) = 0$ and $f'(0) = 1$. Thus

$$\sin x \approx x \quad \text{for } x \text{ near } 0 \text{ in radians.}$$

For example, $\sin(0.02) \approx 0.02$. This describes the local behavior of sine; it is not an exact identity.

2 Differentials: The Predicted Output Change

2.1 Actual change versus tangent-line change

Suppose $y = f(x)$ and the input changes from a to $a + \Delta x$. The actual output change is

$$\Delta y = f(a + \Delta x) - f(a).$$

The tangent line predicts $\Delta y \approx f'(a)\Delta x$. We give that prediction its own notation: choose $dx = \Delta x$ and define

$$dy = f'(a) dx.$$

The *differential* dy is exactly the change along the tangent line:

$$dy = L(a + dx) - L(a), \quad \Delta y \approx dy.$$

Thus $f(a + dx) \approx f(a) + dy$. The symbol dx denotes a real input increment, which may be positive or negative.

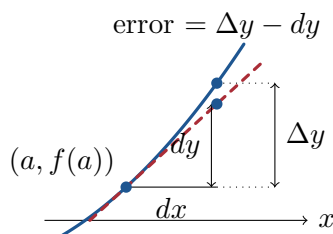


Figure 3: The same input change gives two different output changes. For the growing square at $a = 3$ with $dx = 0.1$,

$$dy = 2(3)(0.1) = 0.6, \quad \Delta y = 0.61.$$

The differential predicts the change; the extra corner accounts for the difference. If $dx = -0.1$, then $dy = -0.6$ and $\Delta y = -0.59$: both indicate a decrease.

2.2 Estimating measurement error

Example 2.1 (An uncertain radius). A sphere's measured radius is 10 cm, with uncertainty ± 0.1 cm. Estimate the resulting volume uncertainty.

For $V = \frac{4}{3}\pi r^3$, the differential is $dV = 4\pi r^2 dr$. At $r = 10$ and $dr = \pm 0.1$,

$$dV = 4\pi(10)^2(\pm 0.1) = \pm 40\pi \text{ cm}^3.$$

The estimated uncertainty is about $40\pi \text{ cm}^3$. Since $dV/V = 3dr/r$, a 1% radius uncertainty gives approximately 3% volume uncertainty. These are linear estimates, not exact error bounds.

3 From Linear Approximations to L'Hôpital's Rule

3.1 What does a zero-over-zero form mean?

If numerator and denominator both approach zero, their quotient depends on how quickly each vanishes. For example, as $x \rightarrow 0$,

$$\lim_{x \rightarrow 0} \frac{2x}{x} = 2, \quad \lim_{x \rightarrow 0} \frac{x^2}{x} = 0, \quad \lim_{x \rightarrow 0} \frac{x^2}{x^4} = +\infty.$$

The form $0/0$ is therefore *indeterminate*: it does not specify a limit.

3.2 Comparing the two tangent lines

Suppose $f(a) = g(a) = 0$, both functions are differentiable at a , and $g'(a) \neq 0$. Their linear approximations are

$$f(x) \approx f'(a)(x - a), \quad g(x) \approx g'(a)(x - a).$$

The common input change cancels, suggesting

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)}.$$

The derivative definition makes this precise:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(x)/(x - a)}{g(x)/(x - a)} = \frac{f'(a)}{g'(a)}.$$

The denominator approaches a nonzero number, so the quotient limit law applies. If the derivatives are also continuous at a , this equals $\lim_{x \rightarrow a} f'(x)/g'(x)$.

Example 3.1 (Comparing slopes at zero). For $f(x) = e^{2x} - 1$ and $g(x) = \ln(1 + 3x)$, the slopes at zero are 2 and 3. Consequently,

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\ln(1 + 3x)} = \frac{2}{3}.$$

The local models are $2x$ and $3x$.

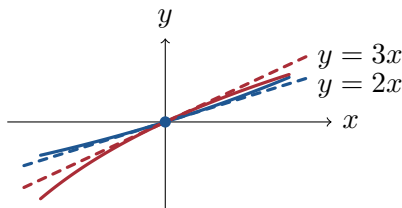


Figure 4: The ratio of the tangent slopes predicts the limiting quotient.

This argument requires $g'(a) \neq 0$. If both tangent slopes vanish, we cannot divide the linear predictions. The general rule below extends the comparison to derivatives at nearby points, including such cases.

3.3 Connecting infinity-over-infinity to zero-over-zero

When f and g tend to infinity, their reciprocals tend to zero. The identity $f/g = (1/g)/(1/f)$ connects the two indeterminate forms, with the reciprocals in reversed order.

Example 3.2 (Using the reciprocal functions). Evaluate $\lim_{x \rightarrow 0^+} \frac{1/x}{1/(3x + x^2)}$.

Set $f(x) = 1/x$ and $g(x) = 1/(3x + x^2)$. Both tend to $+\infty$ as $x \rightarrow 0^+$. Taking reciprocals gives

$$\frac{f(x)}{g(x)} = \frac{3x + x^2}{x}.$$

The numerator has linearization $3x$ at zero; the denominator is already x . Their ratio therefore approaches 3.

Now compare the derivatives of the *original* functions:

$$f'(x) = -\frac{1}{x^2}, \quad g'(x) = -\frac{3 + 2x}{(3x + x^2)^2}.$$

Their quotient simplifies to

$$\frac{f'(x)}{g'(x)} = \frac{(3x + x^2)^2}{x^2(3 + 2x)} = \frac{(3 + x)^2}{3 + 2x}.$$

Therefore,

$$\lim_{x \rightarrow 0^+} \frac{f'(x)}{g'(x)} = \frac{9}{3} = 3.$$

Thus the reciprocal linearizations and the original derivative quotient give the same limit.

If the derivative quotient remains indeterminate, check the hypotheses and apply the rule again. Otherwise, evaluate its limit directly; it may be finite or infinite. We state the general rule without its full proof.

3.4 L'Hôpital's rule

Theorem 3.1 (L'Hôpital's rule). Let x approach a , where a may also denote $+\infty$ or $-\infty$. At a finite endpoint, interpret the limits as one-sided. Suppose f, g are differentiable on the approaching interval or tail, with $g(x) \neq 0$ and $g'(x) \neq 0$. If the quotient has form $0/0$ or infinity-over-infinity, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists as a real number or as $\pm\infty$. Both limits use the same approach. For a two-sided limit, the two one-sided results must agree.

Differentiate the numerator and denominator separately. This is a comparison of limits, not the quotient rule and not an identity between the two quotients.

4 Evaluating Indeterminate Quotients

Check the indeterminate form before each application.

4.1 Zero-over-zero examples

Example 4.1 (One application). Evaluate $\lim_{x \rightarrow 0} \frac{\ln(1+4x)}{x}$.

Both numerator and denominator approach zero. The denominator derivative is 1, so

$$\lim_{x \rightarrow 0} \frac{\ln(1+4x)}{x} = \lim_{x \rightarrow 0} \frac{4/(1+4x)}{1} = 4.$$

This agrees with the local prediction $\ln(1+4x) \approx 4x$.

Example 4.2 (Repeated application). Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

The original quotient has form 0/0. Its derivative quotient also has form 0/0, so apply the rule twice:

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2}.$$

The denominator derivatives $2x$ and 2 are nonzero for $x \neq 0$ near zero. Here both original tangent slopes vanish, so the single linearization at zero is insufficient.

4.2 Infinity-over-infinity examples

Example 4.3 (Logarithmic growth). Both $\ln x$ and $\sqrt{x}$ tend to $+\infty$ as $x \rightarrow \infty$. Thus

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

The logarithm grows more slowly than the square root.

Example 4.4 (Polynomial and exponential growth). Both x^2 and e^x tend to $+\infty$ as $x \rightarrow \infty$. The first derivative quotient still has this form:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0.$$

The denominator derivative remains $e^x > 0$ at each step. Exponential growth makes x^2 negligible by comparison.