Calculus with Analytic Geometry

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1 Key Topics

Today we review polynomial and rational functions.

2 Polynomial Functions

A polynomial is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where $a_0, a_1, \ldots, a_n$ are constants known as coefficients and the integer n is the degree of the polynomial.

The roots of a polynomial are values of x for which $f(x) = 0$. In general, a polynomial of degree n has exactly n roots (real or complex). If a root is a real number, then that root is a x-intercept of the polynomial, that is, the graph of $f(x)$ intersect the x-axis at that point. For example, the polynomial $f(x) = x^2 - 2x - 3$ shown in Figure 1 has roots $x = -1, 3$.

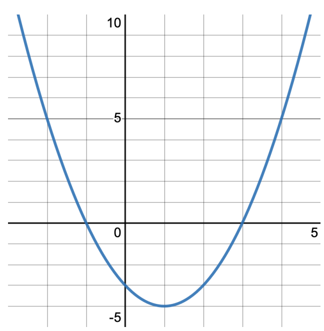

Figure 1: Graph of the polynomial $f(x) = x^2 - 2x - 3$

2.1 Quadratic Formula

If the polynomial is a quadratic (has degree 2), then we can find the roots of the polynomial using the quadratic formula. In particular, the roots of the quadratic $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For example, the roots of the polynomial $f(x) = x^2 - 2x - 3$ are given by

$$\frac{2 \pm \sqrt{4 - 4(1)(-3)}}{2(1)} = \frac{2 \pm \sqrt{16}}{2} = -1, 3.$$

2.2 Completing the Square

Sometimes it is more efficient to complete the square in order to find the roots of a polynomial. For example, $f(x) = x^2 + 2x + 1$ is a perfect square since it can be written as

$$f(x) = (x + 1)^2.$$

Therefore, the roots of $f(x)$ are $-1, -1$. This strategy is effective even for polynomials that are not perfect squares. For example, the polynomial $f(x) = x^2 - 2x - 3$ can be written as

$$f(x) = (x - 1)^2 - 4.$$

Therefore, the roots of $f(x)$ satisfy $(x - 1)^2 = 4$, that is, $x = 1 \pm 2 = -1, 3$.

2.3 Factoring

Suppose that the polynomial $f(x)$ has roots $r_1, r_2, \ldots, r_n$. Then, $f(x)$ can be written as a product of linear factors as follows

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_n).$$

As an example, note that

$$f(x) = x^2 - 2x - 3 = (x + 1)(x - 3).$$

When factoring a polynomial with real coefficients, we may choose to keep quadratic factors when dealing with complex roots. For example,

$$x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2),$$

which can't be reduced further over the real numbers since the roots of $x^2 + 1$ are $\pm i$.

3 Rational Functions

A rational function is a quotient of two polynomial functions. For example, the rational function $f(x) = \frac{x^2-x-2}{x^2-1}$ shown in Figure 2

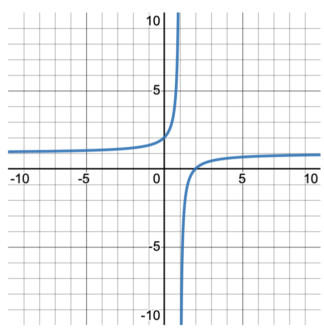

Figure 2: Graph of the rational function $f(x) = \frac{x^2-x-2}{x^2-1}$

The roots of a rational function $f(x)$ are values of x such that $f(x) = 0$. The vertical asymptotes of a function $f(x)$ are vertical lines of the form $x = a$ such that the graph of $f(x)$ is unbounded as x approaches a. From the graph of $f(x) = \frac{x^2-x-2}{x^2-1}$ shown in Figure 2, it is clear that the function has a vertical asymptote at $x = 1$ and a root at $x = 2$.

4 Exercises

a. Factor the numerator and denominator of $f(x) = \frac{x^2-x-2}{x^2-1}$ and use the factorization to determine all roots and vertical asymptotes of $f(x)$.

b. Use polynomial long division to fully simplify the rational function

$$f(x) = \frac{x^3 - 2x^2 + 2x - 1}{x - 1}.$$

c. Use partial fraction decomposition to rewrite

$$f(x) = \frac{x - 1}{x^2 + 5x + 6}$$

as the sum of two rational functions with linear denominators.