Calculus with Analytic Geometry

Thomas R. Cameron

January 10, 2024

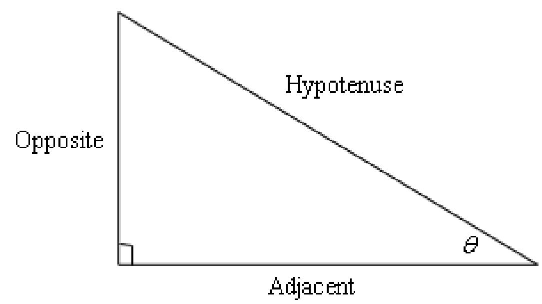
1 Key Topics

Today we review trig functions.

2 Trig Functions

The six standard trig functions are defined over the right triangle

$$\begin{aligned}
\sin(\theta) &= \frac{\text{Opposite}}{\text{Hypotenuse}}\\
\cos(\theta) &= \frac{\text{Adjacent}}{\text{Hypotenuse}}\\
\tan(\theta) &= \frac{\text{Opposite}}{\text{Adjacent}}\\
\csc(\theta) &= \frac{\text{Hypotenuse}}{\text{Opposite}}\\
\sec(\theta) &= \frac{\text{Hypotenuse}}{\text{Adjacent}}\\
\cot(\theta) &= \frac{\text{Adjacent}}{\text{Opposite}}
\end{aligned}$$



From these definitions, we can derive some important identities. To this end, let O denote the opposite side, A denote the adjacent side, and H the hypotenuse side of the right triangle. For example, consider the Pythagorean Identity:

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2\\
&= \frac{O^2 + A^2}{H^2}\\
&= \frac{H^2}{H^2} = 1.
\end{aligned}$$

By dividing the Pythagorean Identity by $\sin^2(\theta)$ or $\cos^2(\theta)$, we obtain the following identities

$$\begin{aligned}
1 + \cot^2(\theta) &= \csc^2(\theta),\\
1 + \tan^2(\theta) &= \sec^2(\theta).
\end{aligned}$$

2.1 Unit Circle

By embedding right triangles into the unit circle (circle of radius 1 centered at the origin), we are able to identify points on the circle using cosine and sine functions. For example, see Figure 1 where we have a right triangle with hypotenuse 1 embedded in the 1st quadrant of the unit circle such that the adjacent side lies

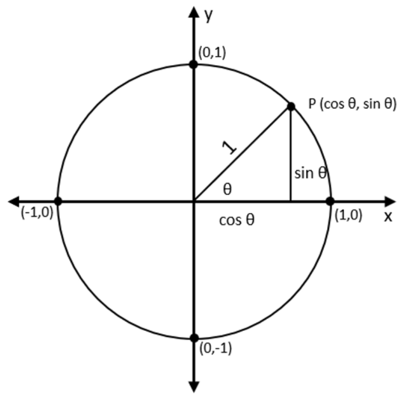


Figure 1: Unit circle with right triangle embedded inside the 1st quadrant

on the x-axis and the opposite side is parallel to the y-axis. Hence, the points on the unit circle in the 1st quadrant can be identified by $(\cos(\theta), \sin(\theta))$ as defined on the right triangle.

We extend the definitions of $\cos(\theta)$ and $\sin(\theta)$ for all angles of θ using the points on the unit circle. For example, the sine (red) and cosine (blue) functions are shown over the domain $[-\pi, \pi]$ in Figure 2.

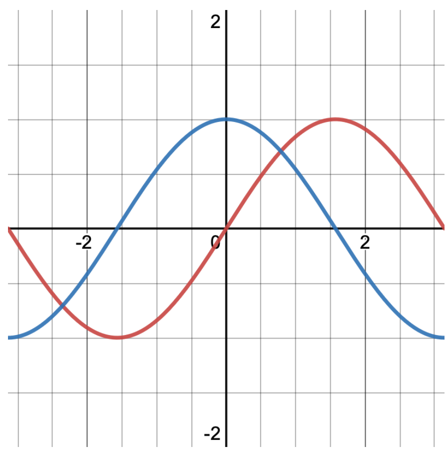


Figure 2: Sine (red) and Cosine (blue) functions

2.2 Inverse Trig Functions

The six standard trig functions map an angle θ to a ratio. The inverse trig functions map a ratio to an angle θ. For example,

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \leftrightarrow \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \leftrightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$
$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \leftrightarrow \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

3 Exercises

Let $y = \arcsin(x)$, where $0 < y < \pi/2$. Then, draw a right triangle with angle y, opposite side x, and hypotenuse 1. Then, identify the following values in terms of x.

a. $\cos(y)$

b. $\tan(y)$

c. $\csc(y)$