Calculus with Analytic Geometry

Thomas R. Cameron

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1 Key Topics

Today we review log and exponential functions.

2 Exponential Functions

The exponential function with base $b > 0$ is defined by $f(x) = b^x$. The long term behavior of an exponential function varies depending on the value of b. In particular, if $0 < b < 1$, then $f(x)$ approaches zero as x gets bigger; if $b = 1$, then $f(x) = 1$ for all x; if $b > 1$, then $f(x)$ approaches infinity as x gets bigger.

In calculus, a natural choice for the base of an exponential function is Euler's number:

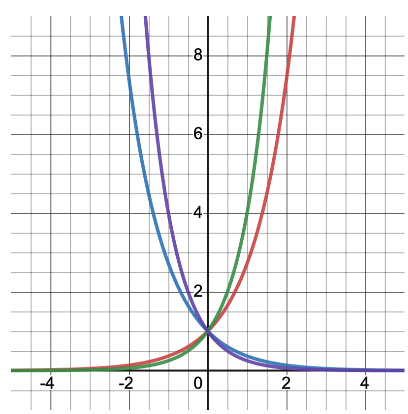
$$e = 2.71828\ldots.$$

This number was first introduced by Jacob Bernoulli in 1683 using a hypothetical financial transaction: Imagine you invest 1 dollar at 100% annual interest. Given that your investment is compounded $n \geq 1$ times, Table 1 shows the value of your investment at the end of the year. As n gets bigger, the value of your investment approaches the number e.

n	Value
1	2
2	2.25
3	$2.37037\ldots$
4	$2.44140\ldots$
$\vdots$	$\vdots$
10	$2.59374\ldots$
$\vdots$	$\vdots$
100	$2.70481\ldots$
$\vdots$	$\vdots$

Table 1: Hypothetical financial transaction proposed by Jacob Bernoulli

In the figure on the right, we display the exponential functions e^x (red), 4^x (green), e^{-x} (blue), and 4^{-x} (purple). Note that the graph of each of the exponential functions includes the point $(0, 1)$.



In general, exponential functions can be written in the form $f(x) = ab^x$, where $b > 0$ is the base and a is an arbitrary constant. A common algebraic problem is to determine constants a and b given two points on the graph.

3 Logarithmic Functions

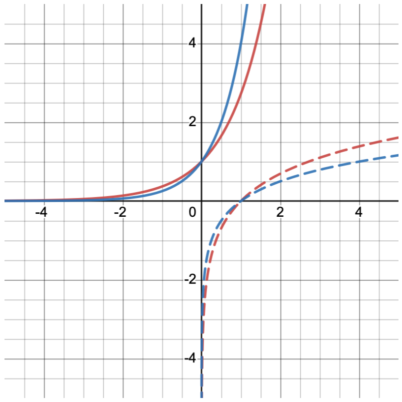
The exponential function $f(x) = b^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$. Moreover, over this domain, the exponential function $f(x)$ is one-to-one (passes the horizontal line test); hence, $f(x)$ has an inverse. By definition, the inverse function $g(x)$ satisfies

$$y = f(x) \;\Leftrightarrow\; x = g(y).$$

Hence, the inverse function $g(x)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.

In particular, the inverse of the exponential function $f(x) = b^x$ is known as the log base b function and is denoted by $g(x) = \log_b(x)$. In the case of the natural base e, the inverse function is known as the natural logarithm function and is denoted by $g(x) = \ln(x)$.

In the figure on the right, we display the exponential functions e^x (red), 4^x (blue) and their corresponding inverse functions $\ln(x)$ (dashed-red), $\log_4(x)$ (dashed-blue). Note that the graph of each of the exponential functions includes the point $(0, 1)$ and the graph of each of the logarithmic functions includes the point $(1, 0)$.



4 Properties

When working with exponential or logarithmic functions, it is important to be aware of the following properties:

1. $b^0 = 1$
2. $b^x b^y = b^{x+y}$
3. $(b^x)^y = b^{xy}$
4. $\frac{b^x}{b^y} = b^{x-y}$

1. $\log_b(1) = 0$
2. $\log_b(xy) = \log_b(x) + \log_b(y)$
3. $\log_b(x^y) = y \log_b(x)$
4. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

It is worth emphasizing that the log properties can be derived from the exponential properties. For example, by definition of the inverse,

$$b^0 = 1 \;\Leftrightarrow\; \log_b(1) = 0.$$

As another example, we will show that exponential property 2 implies logarithmic property 2. To this end, let $x = b^w$ and $y = b^z$. Then,

$$\begin{aligned}
\log_b(xy) &= \log_b\left(b^w b^z\right) \\
&= \log_b\left(b^{w+z}\right) \\
&= w + z \\
&= \log_b\left(b^w\right) + \log_b\left(b^z\right) = \log_b(x) + \log_b(y).
\end{aligned}$$

5 Exercises

a. Identify the exponential function of the form $y = ab^x$ whose graph intersections the points $(0, 2)$ and $(3, 54)$.

b. Solve the following log/exponential equations

- $8e^x - 12 = 7$
- $e^{x-3} = 5$
- $\ln(x - 2)^2 = 12$
- $\ln(x) + \ln(x - 2) = \ln(3)$