Calculus with Analytic Geometry

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1 Key Topics

Today we conclude our review of log and exponential functions, and we introduce the limit of a function.

2 Log and Exponential Functions

Recall the exponential and log properties shown below.

1. $b^0 = 1$
2. $b^x b^y = b^{x+y}$
3. $(b^x)^y = b^{xy}$
4. $\frac{b^x}{b^y} = b^{x-y}$

1. $\log_b(1) = 0$
2. $\log_b(xy) = \log_b(x) + \log_b(y)$
3. $\log_b(x^y) = y\log_b(x)$
4. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

It can be shown that the exponential and log properties are equivalent. For example, by definition of the inverse,

$$b^0 = 1 \iff \log_b(1) = 0.$$

As another example, we will show that exponential property 2 implies logarithmic property 2. To this end, let $x = b^w$ and $y = b^z$. Then,

$$\begin{aligned}
\log_b(xy) &= \log_b\left(b^w b^z\right) \\
&= \log_b\left(b^{w+z}\right) \\
&= w + z \\
&= \log_b\left(b^w\right) + \log_b\left(b^z\right) = \log_b(x) + \log_b(y).
\end{aligned}$$

Finally, we will solve the following log equation:

$$\ln(x) + \ln(x-2) = \ln(3).$$

3 Limits of Functions

The limit of a function f at c is denoted by

$$\lim_{x\to c} f(x).$$

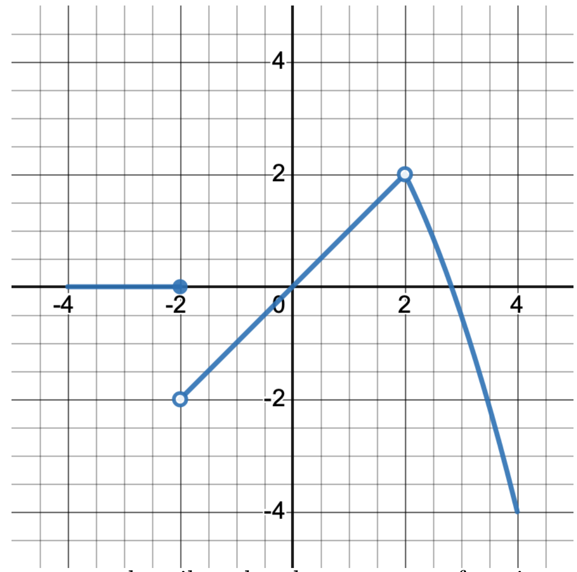
If it exists, the limit is the value that the function approaches as its input approaches, but is not equal to, c.

Consider the piecewise defined function

$$f(x) = \begin{cases} 0 & -4 \le x \le -2 \\ x & -2 < x < 2 \\ 4 - \frac{1}{2}x^2 & 2 < x \le 4 \end{cases}$$

Note the following limits:

$$\lim_{x \to -2} f(x) = DNE \qquad \lim_{x \to 2} f(x) = 2.$$



The limit is an essential concept in calculus as it allows us to describe what happens to a function as its input approaches a value (or infinity). For example, the function f shown above is undefined at $x = 2$. Hence, it makes no sense to discuss the value of $f(2)$. However, we can discuss the value of f as x approaches, but is not equal to, 2.

One sided limits also play an important role. The right (left) sided limits of a function f at c are denoted (respectively) by

$$\lim_{x \to c^+} f(x) \qquad \lim_{x \to c^-} f(x)$$

and is equal to the value the function approaches as its input approaches, but is not equal to, c from the right (left). Note that if the limit exists then

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x).$$

Consider the piecewise defined function

$$f(x) = \begin{cases} 8 + 2x & -4 \le x < -2 \\ x^2 & -2 < x \le 2 \\ \frac{1}{x-2} & x > 2 \end{cases}$$

Note the following limits:

$$\begin{aligned} &\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = \lim_{x \to -2} f(x) = 4 \\ &\lim_{x \to 2^-} f(x) = 4 \\ &\lim_{x \to 2^+} f(x) = +\infty \\ &\lim_{x \to +\infty} f(x) = 0 \end{aligned}$$