Calculus with Analytic Geometry

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1 Key Topics

Today we introduce techniques for evaluating limits analytically.

2 Properties of Limits

The following proposition states the basic limits.

Proposition 2.1. *Let b and c be real numbers. Then,*

$$\lim_{x\to c} b = b \qquad \lim_{x\to c} x = c.$$

The next proposition states the properties of limits.

Proposition 2.2. *Let b and c be real numbers and let f and g be functions with the limits $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$. Then, the following properties hold:*

I. $\lim_{x\to c} (bf(x)) = bL$,

II. $\lim_{x\to c} (f(x) + g(x)) = L + K$,

III. $\lim_{x\to c} (f(x)g(x)) = LK$,

IV. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K},\ K \neq 0$.

We can use the limit properties (Proposition 2.2) to build upon the basic limits (Proposition 2.1) to evaluate the limits of more complicated functions.

2.1 Polynomial Functions

We begin with an example.

Example 2.3. Let $f(x) = 3 + 2x - x^2$ and let c be a real number. Note that

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} \left(3 + 2x - x^2\right) \\
&= \lim_{x\to c}(3) + \lim_{x\to c}(2x) + \lim_{x\to c}(-x^2) \\
&= 3 + 2\lim_{x\to c}(x) - \lim_{x\to c}(x \cdot x) \\
&= 3 + 2c - c^2 = f(c).
\end{aligned}$$

We can extend the result in Example 2.3 to all polynomials.

Proposition 2.4. *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree n. Then,*

$$\lim_{x\to c} f(x) = f(c).$$

2.2 Rational Functions

We begin with an example.

Example 2.5. Consider the limit

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+x+2}{x+1} &= \frac{\lim_{x\to 1}(x^2+x+2)}{\lim_{x\to 1}(x+1)}\\ &= \frac{4}{2} = 2.\end{aligned}$$

At first glance, it may appear that we can evaluate the limit of a rational function by simply evaluating the rational function at that point (just as we can with polynomial functions). However, if the denominator is equal to zero at that point, then we must take extra care when evaluating the limit.

Example 2.6. Consider the limit

$$\begin{aligned}\lim_{x\to -1}\frac{x^2+x+2}{x+1} &= \frac{\lim_{x\to -1}(x^2+x+2)}{\lim_{x\to -1}(x+1)}\\ &= \frac{2}{\lim_{x\to -1}(x+1)} = \pm\infty.\end{aligned}$$

This limit indicates that $x=-1$ is a vertical asymptote of the rational function $\frac{x^2+x+2}{x+1}$. We can specify the behavior of this vertical asymptote by considering the right and left sided limits:

$$\begin{aligned}\lim_{x\to -1^+}\frac{x^2+x+2}{x+1} &= \frac{2}{\lim_{y\to 0^+} y} = +\infty,\\ \lim_{x\to -1^-}\frac{x^2+x+2}{x+1} &= \frac{2}{\lim_{y\to 0^-} y} = -\infty,\end{aligned}$$

where the substitution $y=x+1$ was made.

Example 2.7. Consider the limit

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-1}.$$

Since both the numerator and denominator go to zero in the limit, we have a common factor of $(x-1)$. When taking the limit, we never consider what happens at $x=1$, hence we can cancel this common factor out and then take the limit:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{x+2}{x+1} = \frac{3}{2}.$$

This limit indicates that there is a hole in the graph of $\frac{x^2+x-2}{x^2-1}$ at the point $(1,3/2)$.