Calculus with Analytic Geometry

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1 Key Topics

Last time we used the properties of limits to show that the limit of a polynomial can be found by evaluating the polynomial directly. Today, we review algebraic techniques for evaluating limits that are initially in the indeterminate form 0/0.

1.1 Factoring

Consider the following limit

$$\lim_{x\to 2}\frac{x^2+2x-8}{x-2}.$$

Evaluating the numerator and denominator directly, we see that the above limit is in the indeterminate form 0/0. Hence, we factor the numerator and cancel out the common factor before evaluating the limit:

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+2x-8}{x-2} &= \lim_{x\to 2}\frac{(x+4)(x-2)}{x-2}\\ &= \lim_{x\to 2}(x+4) = 6.\end{aligned}$$

This factoring technique can work with some exponential expressions as well; for example, consider the following limit:

$$\begin{aligned}\lim_{x\to 0}\frac{3^{2x}-1}{3^x-1} &= \lim_{x\to 0}\frac{(3^x-1)(3^x+1)}{3^x-1}\\ &= \lim_{x\to 0} 3^x+1 = 2.\end{aligned}$$

1.2 Long Division

Consider the following limit

$$\lim_{x\to 2}\frac{x^3-8}{x-2}.$$

Once again, the limit is in the indeterminate form 0/0. Using long division, we find that

$$x^3-8=(x-2)(x^2+2x+4).$$

Therefore, we have

$$\begin{aligned}\lim_{x\to 2}\frac{x^3-8}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}\\ &= \lim_{x\to 2}(x^2+2x+4) = 12.\end{aligned}$$

1.3 Rationalization Technique

Consider the following limit

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

Once again, the limit is in the indeterminate form 0/0. This time, we must use a rationalization technique before evaluating the limit

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \\
&= \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+1}+1\right)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}.
\end{aligned}$$

1.4 Squeeze Theorem

Transcendental functions can be more difficult to work with in the presence of the indeterminate form 0/0 since they are not easily factored. For example, consider the following limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

Using the right triangle embedded in the unit circle, one can show that

$$\cos(x) \le \frac{\sin(x)}{x} \le 1, \; -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Therefore, we have

$$\begin{aligned}
\lim_{x\to 0} \cos(x) \le \lim_{x\to 0} \frac{\sin(x)}{x} \le \lim_{x\to 0} 1 \\
1 \le \lim_{x\to 0} \frac{\sin(x)}{x} \le 1,
\end{aligned}$$

and it follows that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

2 Exercises

Evaluate each of the following limits:

a. $\lim_{x\to 1} \frac{x^2+3x-4}{x-1}$

b. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

c. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$