Calculus with Analytic Geometry

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1 Key Topics

Last week, we introduced the concept of the limit of a function, and we learned how to evaluate the limit graphically and analytically. Today, we use the limit of a function to define continuity.

1.1 Continuity

We say that a function $f(x)$ is *continuous* at c provided that

I. $f(c)$ is defined,

II. $\lim_{x\to c} f(x)$ exists,

III. $\lim_{x\to c} f(x) = f(c)$.

For example, we have seen that all polynomial functions $p(x)$ satisfy parts I-III in the above definition, for any c; hence, all polynomial functions are continuous on their entire domain. Furthermore, we can extend this result to rational functions $f(x) = \frac{p(x)}{q(x)}$ using limit properties. In particular, $f(x)$ is continuous for all c such that $q(c) \neq 0$.

It is important to note that we may slightly modify the definition of continuity if we are discussing a value c that is an endpoint of the domain of $f(x)$. For example, $\sqrt{x}$ is continuous at 0 since

I. $\sqrt{0} = 0$,

II. $\lim_{x\to 0^+} \sqrt{x}$ exists,

III. $\lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$.

Finally, we will often use our knowledge of the properties and graphs of radical and transcendental functions to determine their limiting value. For example,

$$\lim_{x\to c} \sqrt{x} = \sqrt{c},\ c \geq 0$$
$$\lim_{x\to c} \sin(x) = \sin(x),\ -\infty < c < \infty$$
$$\lim_{x\to c} e^x = e^x,\ -\infty < c < \infty$$
$$\lim_{x\to c} \ln(x) = \ln(c),\ c > 0.$$

1.2 Types of Discontinuity

A function $f(x)$ is discontinuous at c if any part of the continuity definition does not hold; in fact, each part of the definition leads to a different type of discontinuity.

1.2.1 Holes and Asymptotes

A hole in the graph of a function occurs at $x = c$ if $f(c)$ is undefined but $\lim_{x\to c} f(x)$ exists and is finite. A vertical asymptote occurs if $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$.

Example 1.1. The function

$$f(x) = x\sin\left(\frac{1}{x}\right)$$

has a hole at $x = 0$. The function

$$f(x) = \begin{cases} x & x \leq 0 \\ \frac{1}{x} & x > 0 \end{cases}$$

has a vertical asymptote at $x = 0$.

1.2.2 Jump Discontinuities

A jump discontinuity occurs at $x = c$ if $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+}$ both exist and are finite but $\lim_{x\to c} f(x)$ does not exist, i.e., the left and right sided limits are not equal.

Example 1.2. The function

$$f(x) = \begin{cases} x & x \leq 0 \\ x^2 + 1 & x > 0 \end{cases}$$

has a jump discontinuity at $x = 0$.

1.2.3 Removable Discontinuities

A removable discontinuity occurs at $x = c$ if $\lim_{x\to c} f(x)$ exists and $f(c)$ exists but $f(x)$ is not continuous at c, i.e., the limiting value and the function value are not equal.

Example 1.3. The function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 2 & x = 0 \end{cases}$$

has a removable discontinuity at $x = 0$. Note that removable discontinuities can be "repaired" by changing the function value definition to match the limiting value of the function. For instance, if we define $f(0) = 0$, then the above function is continuous.

1.3 Intermediate Value Theorem

One important application of the concept of continuity is the intermediate value theorem.

Theorem 1.4 (Intermediate Value Theorem)**.** *Suppose that f is continuous on $[a, b]$, $f(a) \neq f(b)$, and k is any value between $f(a)$ and $f(b)$. Then, there exists a c between a and b such that $f(c) = k$.*

Example 1.5. Consider the polynomial

$$p(x) = x^3 + 2x^2 + 2x - 1.$$

Note that $f(0) < 0$ and $f(1) > 0$; hence, there exists a c between 0 and 1 such that $f(c) = 0$.

2 Exercises

Identify all discontinuities (and types) of the following function

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x < 0 \\ 2 + x & 0 \leq x \leq 2 \\ 1 + x^2 & 2 < x \leq 3 \\ \frac{1}{x-3} & x > 3 \end{cases}$$