Calculus with Analytic Geometry

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1 Key Topics

Today, we introduce secant lines, tangent lines, and the derivative.

1.1 Secant and Tangent Lines

Let f be a function and (x_0, y_0) and (x_1, y_1) denote points on the graph of f. Then, the secant line connecting the two points is given by

$$y - y_0 = m(x - x_0),\ m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The tangent line of f at (x_0, y_0) is defined by

$$y - y_0 = m(x - x_0),\ m = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

One can visual the tangent line by moving the point (x_1, y_1) on the secant line closer and closer to the point (x_0, y_0).

Example 1.1. Let $f(x) = 1 - x^2$. The tangent line of f at $(1, 0)$ has the form

$$y - 0 = m(x - 1),$$

where

$$\begin{aligned} m &= \lim_{\Delta x \to 0} \frac{[1 - (1 + \Delta x)^2] - [1 - 1^2]}{\Delta x \to 0} \\ &= \lim_{\Delta x \to 0} \frac{1 - (1 + 2\Delta x + \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-2\Delta x - \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-2 - \Delta x) = -2. \end{aligned}$$

1.2 The Derivative

Let f be a function. The derivative of f at x is denoted by $f'(x)$ and is defined by the slope of the tangent line of f at $(x, f(x))$, i.e.,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Example 1.2. Let $f(x) = 1 - x^2$. Then, the derivative of f is

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[1-(x+\Delta x)^2] - [1-x^2]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[1-(x^2+2x\Delta x+\Delta x^2] - [1-x^2]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x. \end{aligned}$$

2 Exercises

Find the derivative of each function below:

a. $f(x) = 3x + 4$

b. $f(x) = x^2$

c. $f(x) = \sqrt{x}$

d. $f(x) = \frac{1}{x}$