Calculus with Analytic Geometry

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1 Key Topics

Today, we continue our discussion of the derivative and use its limit definition to derive basic rules for evaluating the derivative of a function.

Recall that the derivative of a function f is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

provided the above limit exists. If the limit exists for a single value of x, then we say that f is differentiable at x. If the limit exists for all $x \in [a, b]$, then we say that f is differentiable on $[a, b]$.

Example 1.1. Let $f(x) = \sqrt{x}$. Then,

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x \left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Note that $f'(x) = \frac{1}{2\sqrt{x}}$ has domain $(0, \infty)$. Hence, $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$.

1.1 Basic Derivative Rules

Since the derivative of a function is defined by a limit, we can use the limit properties to derive several basic rules for evaluating the derivative. When describing these rules, it is convenient to use Leibniz's notation:

$$\frac{d}{dx} f(x) = f'(x).$$

Proposition 1.2. *Let f and g be differentiable at x, and let c denote a constant. Then, the following properties hold:*

a. $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$

b. $\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x)$

Example 1.3. Let $f(x) = 3x^2 + 4\sqrt{x} - \frac{1}{x}$. Then,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3x^2\right) + \frac{d}{dx}\left(4\sqrt{x}\right) + \frac{d}{dx}\left(-\frac{1}{x}\right) \\ &= 3\frac{d}{dx}\left(x^2\right) + 4\frac{d}{dx}\left(\sqrt{x}\right) - \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= 3(2x) + 4\left(\frac{1}{2\sqrt{x}}\right) - \left(\frac{-1}{x^2}\right) \\ &= 6x + \frac{2}{\sqrt{x}} + \frac{1}{x^2} \end{aligned}$$

1.2 Power Rule

The power rule tells us how to evaluate functions of the form x^n, where n is any real number.

Proposition 1.4. *Let n be any real number. Then,*

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

2 Exercises

Find the derivative of each function below:

a. $f(x) = x^2 + 5 - 3x^{-2}$

b. $f(x) = x^5 - \frac{4}{x^3}$

c. $f(x) = \frac{3x^2+4x-8}{x^{3/2}}$

d. $f(x) = \sqrt{x} - 6x^{1/3}$