Polynomial, Rational, and Inverse Functions

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Math 140: Calculus with Analytic Geometry

Learning objectives

After this lesson, you should be able to:

- identify polynomial and rational functions and state their natural domains;
- factor polynomials and use the zero-product property to solve equations;
- simplify rational expressions while preserving excluded domain values;
- distinguish a removable discontinuity from a vertical asymptote;
- compute compositions and determine their domains;
- decide whether a function is one-to-one; and
- find an inverse function and correctly interchange domain and range.

1 Polynomial functions

Definition. A polynomial function is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer, the coefficients $a_0, \ldots, a_n$ are real numbers, and $a_n \neq 0$. The integer n is the degree of p.

Every polynomial has domain $\mathbb{R}$. Polynomial graphs have no breaks, holes, or vertical asymptotes. In calculus language, polynomials are continuous and differentiable at every real number. These facts will later let us evaluate polynomial limits by direct substitution.

1.1 Factoring and solving equations

The most useful algebraic form depends on the task. Expanded form makes the degree and leading term easy to see; factored form makes the zeros easy to see.

Key principle. If $ab = 0$, then $a = 0$ or $b = 0$. Thus, to solve a polynomial equation, first rewrite it with zero on one side, factor, and then apply the zero-product property.

Example: polynomial equations. Solve each equation over the real numbers.

First consider

$$x^3 - 4x^2 - x + 4 = 0.$$

Factoring by grouping gives

$$\begin{aligned} x^3 - 4x^2 - x + 4 &= x^2(x-4) - 1(x-4) \\ &= (x-4)(x^2-1) \\ &= (x-4)(x-1)(x+1). \end{aligned}$$

The zero-product property therefore gives

$$x = -1,\ 1,\ 4.$$

For the equation $2x^2 + 4x - 3 = 0$, the quadratic formula gives

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} = \frac{-2 \pm \sqrt{10}}{2}.$$

These same algebraic steps arise when locating critical numbers and solving optimization problems.

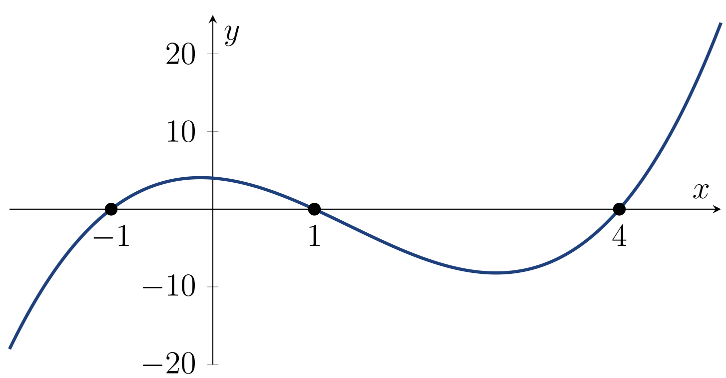

Figure 1. The graph of $p(x) = x^3 - 4x^2 - x + 4$ crosses the x-axis at $x = -1$, $x = 1$, and $x = 4$, matching the three roots found by factoring.

Important factoring patterns include

$$a^2 - b^2 = (a-b)(a+b), \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

and

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

When a quadratic does not factor conveniently, the quadratic formula gives the solutions of $ax^2 + bx + c = 0$, where $a \neq 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

2 Rational functions

Definition. A rational function is a function of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials and q is not the zero polynomial. Its natural domain consists of the real numbers for which $q(x) \neq 0$.

Unlike polynomials, rational functions may be undefined at some real numbers. The first step in nearly every rational-function calculation is therefore to factor the denominator and record the excluded values.

2.1 Cancellation does not restore a domain value

Example: rational expressions and domains. Consider

$$r(x) = \frac{x^2 - x - 6}{x^2 - 4x + 3}.$$

Factoring and simplifying gives

$$r(x) = \frac{(x-3)(x+2)}{(x-1)(x-3)} = \frac{x+2}{x-1}, \qquad x \neq 1, 3.$$

The domain of the original function is

$$(-\infty, 1) \cup (1, 3) \cup (3, \infty).$$

The canceled factor produces a hole at

$$\left(3, \frac{3+2}{3-1}\right) = \left(3, \frac{5}{2}\right),$$

and the factor remaining in the denominator produces the vertical asymptote $x = 1$. Later, limits will make both conclusions precise.

Common error. Cancellation removes a common factor, not a term. For example,

$$\frac{x^2 + 1}{x}$$

cannot be simplified by canceling the x in x^2 with the denominator. Also, a canceled factor does not add its zero back into the domain.

2.2 Holes and vertical asymptotes

Suppose a rational function is written in factored form. A zero of the original denominator has one of two common behaviors:

- If a factor corresponding to that zero remains in the denominator after cancellation, the graph has a vertical asymptote there.
- Otherwise, the graph has a hole there.

The behavior from the preceding example is visible in the following plot.

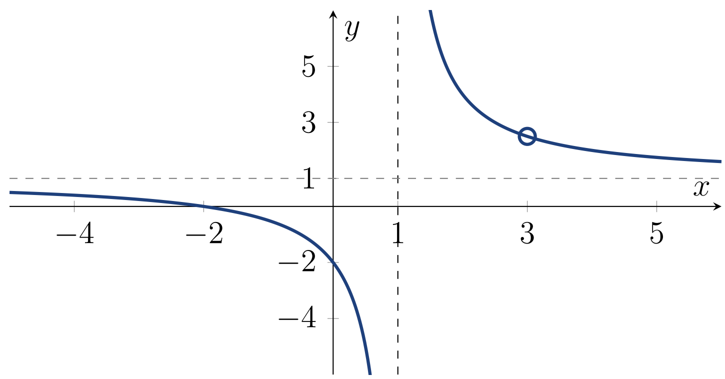

Figure 2. The graph of $r(x) = (x^2 - x - 6)/(x^2 - 4x + 3)$ agrees with $y = (x + 2)/(x - 1)$ except for an open point at $(3, 5/2)$. The dashed lines $x = 1$ and $y = 1$ are its vertical and horizontal asymptotes, respectively.

For end behavior, compare the degrees of the numerator and denominator. If $r(x) = p(x)/q(x)$ is in lowest terms, then:

- if $\deg p < \deg q$, the horizontal asymptote is $y = 0$;
- if $\deg p = \deg q$, the horizontal asymptote is the ratio of the leading coefficients; and
- if $\deg p > \deg q$, polynomial division reveals the relevant polynomial asymptote or end behavior.

These rules describe what happens as $x \to \infty$ or $x \to -\infty$; they do not say that the graph can never cross an asymptote.

3 Composition of functions

Definition. The composition of g with f is

$$(g \circ f)(x) = g(f(x)).$$

The expression is defined only when x is in the domain of f and $f(x)$ is in the domain of g. Order matters: in general, $g \circ f \neq f \circ g$.

Example: composition and domain. Let

$$f(x) = \sqrt{x+1}, \qquad g(x) = \frac{1}{x-2}.$$

Then

$$(f \circ g)(x) = \sqrt{\frac{1}{x-2} + 1} = \sqrt{\frac{x-1}{x-2}}.$$

A sign chart shows that the radicand is nonnegative on $(-\infty, 1] \cup (2, \infty)$, which is the domain of $f \circ g$. In the other order,

$$(g \circ f)(x) = \frac{1}{\sqrt{x+1} - 2}.$$

The square root requires $x \geq -1$, and the denominator is zero at $x = 3$. Thus the domain is $[-1, 3) \cup (3, \infty)$.

Composition is central to the chain rule: differentiating a function such as $(x^2+1)^5$ requires recognizing it as an outer function applied to an inner function.

4 Inverse functions

An inverse function reverses the action of the original function. If $f(a) = b$, then $f^{-1}(b) = a$.

Definition. Functions f and f^{-1} are inverses when

$$f^{-1}(f(x)) = x \quad \text{for every } x \in \mathrm{Dom}(f),$$

and

$$f(f^{-1}(x)) = x \quad \text{for every } x \in \mathrm{Dom}(f^{-1}).$$

Common error. The notation $f^{-1}(x)$ means the inverse function, not the reciprocal:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

in general.

4.1 When does an inverse exist?

A function is one-to-one if different inputs always produce different outputs. Equivalently,

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

A function has an inverse function if and only if it is one-to-one. Graphically, this is the horizontal line test: every horizontal line must meet the graph at most once.

For example, $f(x) = x^2$ is not one-to-one on $\mathbb{R}$ because $f(2) = f(-2)$. Restricting its domain to $[0, \infty)$ makes it one-to-one, and the restricted function has inverse $f^{-1}(x) = \sqrt{x}$.

4.2 Finding an inverse algebraically

To find an inverse:

1. Write $y = f(x)$.
2. Solve the equation for x in terms of y.
3. Interchange the variable names to write $y = f^{-1}(x)$.
4. State the domain of the inverse and verify by composition when useful.

Example: inverse function. Let

$$h(x) = \frac{2x-3}{x+4}.$$

Its domain is $\mathbb{R} \setminus \{-4\}$. Starting with

$$y = \frac{2x-3}{x+4},$$

we solve for x:

$$\begin{aligned} yx + 4y &= 2x - 3, \\ x(y-2) &= -(3+4y), \\ x &= \frac{3+4y}{2-y}. \end{aligned}$$

Therefore,

$$h^{-1}(x) = \frac{4x+3}{2-x}, \qquad x \neq 2.$$

The domain of h^{-1} is the range of h, namely $\mathbb{R} \setminus \{2\}$, and the range of h^{-1} is the domain of h, namely $\mathbb{R} \setminus \{-4\}$. We can verify the formula by composition:

$$h(h^{-1}(x)) = \frac{2\left(\frac{4x+3}{2-x}\right) - 3}{\left(\frac{4x+3}{2-x}\right) + 4} = \frac{11x/(2-x)}{11/(2-x)} = x.$$

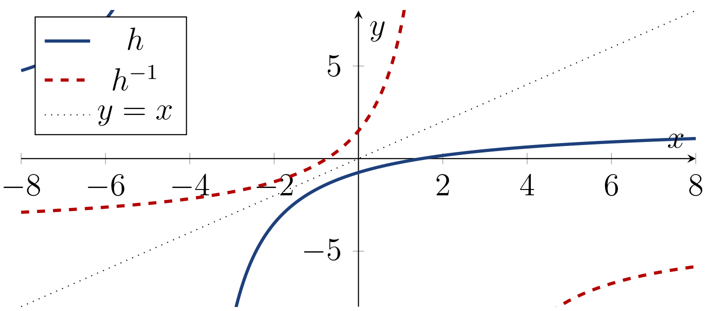

Figure 3. The solid graph of $h(x) = (2x - 3)/(x + 4)$ and the dashed graph of $h^{-1}(x) = (4x + 3)/(2 - x)$ are reflections across the dotted line $y = x$. Reflection swaps the asymptote pairs $x = -4$, $y = -4$ and $y = 2$, $x = 2$.

The graphs of f and f^{-1} are reflections of one another across the line $y = x$: if (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}. This swaps the domain and range:

$$\mathrm{Dom}(f^{-1}) = \mathrm{Range}(f), \qquad \mathrm{Range}(f^{-1}) = \mathrm{Dom}(f).$$

5 Calculus connections

The ideas in this review recur throughout Calculus I:

- **Limits and continuity:** factoring can expose a removable discontinuity, while a remaining denominator zero may signal an infinite limit and vertical asymptote.
- **Derivatives:** polynomial and rational expressions are the basic inputs for the power, product, quotient, and chain rules.
- **Related rates and optimization:** formulas are often polynomial or rational, and solving the resulting equations requires careful factoring and domain checks.
- **Inverse functions:** logarithmic and inverse trigonometric functions are defined as inverses, and their derivative formulas depend on this relationship.
- **Integration:** algebraic simplification and partial fractions later allow certain rational functions to be integrated.

Lesson summary

Polynomials are defined and smooth everywhere. Rational functions inherit their domain restrictions from their original denominators, even after cancellation. Factoring distinguishes holes from vertical asymptotes and supports the limit calculations that begin the course. Composition requires both correct order and careful domain tracking. Finally, only one-to-one functions have inverses; an inverse reverses inputs and outputs, so its domain and range are swapped.