Product and Quotient Rules

Math 140: Calculus with Analytic Geometry

Key Topics

- Product Rule
- Quotient Rule
- Efficient differentiation of rational functions
- Common pitfalls

1 Motivation

Many functions encountered in applications are formed by multiplying or dividing simpler functions. The linearity rules from the previous lecture are not sufficient for these cases, so we introduce two new derivative rules.

2 Product Rule

Theorem 2.1 (Product Rule)**.** *If $f(x) = u(x)v(x)$, where both u and v are differentiable, then*

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Proof. Fix a. Using the limit definition,

$$f'(a) = \lim_{h\to 0} \frac{u(a+h)v(a+h) - u(a)v(a)}{h}.$$

Add and subtract $u(a+h)v(a)$ in the numerator:

$$\frac{u(a+h)v(a+h) - u(a+h)v(a) + u(a+h)v(a) - u(a)v(a)}{h}.$$

Group terms:

$$= \frac{u(a+h)\big(v(a+h) - v(a)\big)}{h} + \frac{v(a)\big(u(a+h) - u(a)\big)}{h}.$$

Taking limits gives

$$f'(a) = u(a)v'(a) + v(a)u'(a).$$

□

Example 2.1. *Differentiate $f(x) = x^2(x^3 - 1)$.*

Let $u(x) = x^2$ and $v(x) = x^3 - 1$. Then

$$f'(x) = 2x(x^3 - 1) + x^2(3x^2) = 5x^4 - 2x.$$

3 Quotient Rule

Theorem 3.1 (Quotient Rule)**.** *If* $f(x) = \dfrac{u(x)}{v(x)}$ *where* $v(x) \neq 0$*, then*

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}.$$

Proof. Write $f(x) = u(x)\,[v(x)]^{-1}$. By the product rule,

$$f'(x) = u'(x)v(x)^{-1} + u(x)\frac{d}{dx}\big(v(x)^{-1}\big).$$

Using the power rule for negative integers,

$$\frac{d}{dx}\big(v(x)^{-1}\big) = -v'(x)v(x)^{-2}.$$

Substituting,

$$f'(x) = \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{v(x)^2} = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}.$$

□

Example 3.1. *Differentiate* $f(x) = \dfrac{x^2+1}{x}$.

Let $u(x) = x^2 + 1$ and $v(x) = x$. Then

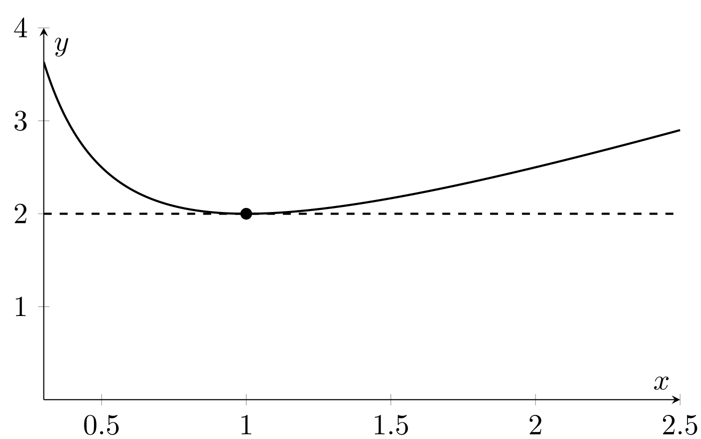
$$f'(x) = \frac{(2x)(x) - (x^2+1)(1)}{x^2} = \frac{x^2-1}{x^2}.$$

4 A Tangent Line Example

Example 4.1. *Find the equation of the tangent line to* $y = \dfrac{x^2+1}{x}$ *at* $x = 1$.

From above, $y' = \dfrac{x^2-1}{x^2}$. At $x = 1$, the slope is 0. Since $y(1) = 2$, the tangent line is

$$y = 2.$$



5 Common Pitfalls

Remark 5.1. *The derivative of a product is* not *the product of the derivatives:*

$$(uv)' \neq u'v'.$$

Remark 5.2. *Although algebraic simplification is sometimes possible, canceling factors before differentiating can change the domain of the function or hide points where the derivative fails to exist.*

6 Why This Matters for Calculus

- Many real-world models involve products and ratios of functions.
- The product and quotient rules are essential for rational functions.
- These rules prepare us for implicit differentiation and related rates.